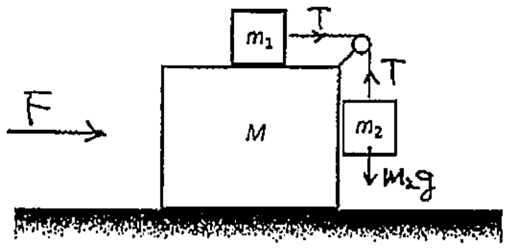
MC Key

1. B
2. D
3. A
4. B
5. E
6. C
7. ~~D~~ (Cancelled)
8. E
9. ~~D~~ (Please refer to solution)
10. B
11. ~~C~~ (Cancelled)
12. A
13. C
14. A
15. E
16. A
17. B
18. C
19. D
20. E

1.

Without relative motion,

$\Rightarrow m_1, m_2$, and M have the same acceleration



Along x-axis:

$$(M + m_1 + m_2)\ddot{x} = F \quad \text{—} \quad (1)$$

$$m_1 \ddot{x} = T \quad \text{—} \quad (2)$$

Along y-axis: $T = m_2 g$ — (3)

(1), (2) and (3) $\Rightarrow$ $(M + m_1 + m_2)\dfrac{m_2 g}{m_1} = F$

Note that m_2 cannot be accelerated horizontally unless the string is inclined at an angle to the vertical, so that it can provide a horizontal force to accelerate m_2.

$T_x = m_2 a$

$T_y = m_2 g$

$\Rightarrow T = \sqrt{T_x^2 + T_y^2} = m_2\sqrt{g^2 + a^2}$

$T = m_1 a \Rightarrow m_1 a = m_2\sqrt{g^2 + a^2} \Rightarrow a = \dfrac{m_2 g}{\sqrt{m_1^2 - m_2^2}}$

$F = (M + m_1 + m_2)a = \dfrac{(M + m_1 + m_2)m_2 g}{\sqrt{m_1^2 - m_2^2}}$

If $m_1 >> m_2$, then

$F \approx \dfrac{(M + m_1 + m_2)m_2 g}{m_1}$

2.

$$\begin{cases} F = bv \\ Fv = 500\,W \end{cases}$$

$$\begin{cases} F = (5\ Ns/m)(v) & \text{——— } 1 \\ Fv = 500\,W & \text{——— } 2 \end{cases}$$

$$\frac{2}{1} \Rightarrow \quad v = \frac{500W}{5\ Ns/m\ (v)}$$

$$v^2 = \frac{500\ N\,ms^{-1}}{5\ N\,s/m}$$

$$v = 10\ ms^{-1}$$

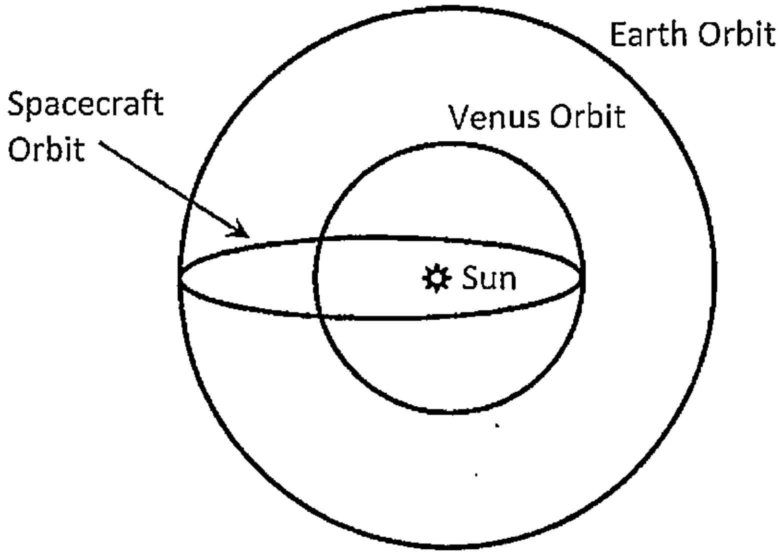
3.

$$\begin{cases} 0.73AU = \dfrac{\lambda(1+\varepsilon)}{1+\varepsilon} = \lambda \\ 1AU = \dfrac{\lambda(1+\varepsilon)}{1-\varepsilon} \end{cases}$$

$$1AU = \frac{0.73AU(1+\varepsilon)}{1-\varepsilon}$$

$$0.27AU = 1.73AU \times \varepsilon$$

$$\Rightarrow \varepsilon = 0.156$$



4.

$$\frac{T_{Earth}^2}{1AU^3} = \frac{T_{spacecraft}^2}{\left(\frac{1AU+0.73AU}{2}\right)^3}$$

$$\Rightarrow T = 0.8 \text{ year}$$

~~$\Rightarrow$ Time Duration = 0.4 year~~

5.

$$\begin{cases} T_1 \cos\alpha + T_2 \cos\beta - mg = 0 \\ T_1 \sin\alpha - T_2 \sin\beta = 0 \end{cases} \Rightarrow T_1 = T_2 \frac{\sin\beta}{\sin\alpha}$$

$$T_2 \frac{\sin\beta}{\sin\alpha} \cos\alpha + T_2 \cos\beta - mg = 0$$

$$T_2 (\sin\beta \cos\alpha + \sin\alpha \cos\beta) = mg \sin\alpha$$

$$T_2 \sin(\alpha + \beta) = mg \sin\alpha$$

$$T_2 = \frac{mg \sin\alpha}{\sin(\alpha+\beta)}$$

$$T_1 = \frac{mg \sin\alpha}{\sin(\alpha+\beta)} \frac{\sin\beta}{\sin\alpha}$$

$$= mg \frac{\sin\beta}{\sin(\alpha+\beta)}$$

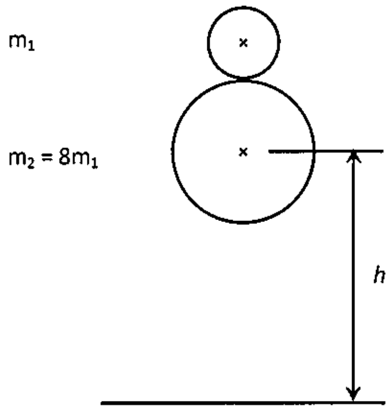
$$\begin{cases} T_1 = m_1 g \\ T_2 = m_2 g \end{cases}$$

$$\Rightarrow \frac{m_1}{m_2} = \frac{T_1}{T_2} = \frac{mg \sin\beta / \sin(\alpha+\beta)}{mg \sin\alpha / \sin(\alpha+\beta)}$$

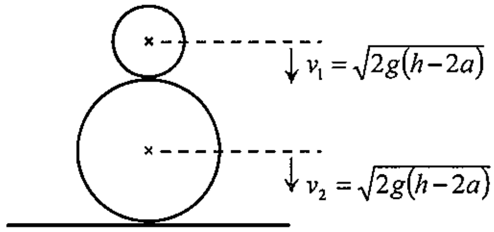
$$= \frac{\sin\beta}{\sin\alpha}$$

7.

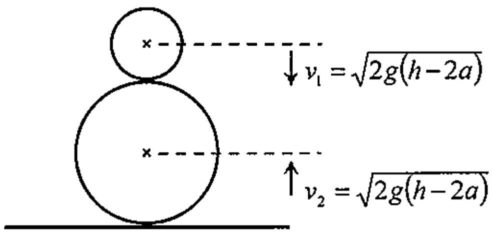
(i) Initial condition



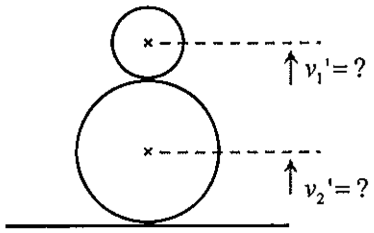
(ii) Just before collision with the ground



(iii) After lower sphere collision from the ground



(iv) Just after elastic collision between spheres



7.

Momentum: $m_2V_2 - m_1V_1 = m_2V_2' + m_1V_1'$ $\uparrow$ +ve $\downarrow$ −ve

$$8m_1V_2 - m_1V_1 = 8m_1V_2' + m_1V_1'$$

$$8V_2 - V_1 = 8V_2' + V_1'$$

$$V_1' = 8V_2 - V_1 - 8V_2'$$

$$= 7V_1 - 8V_2'$$

Energy: $\frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 = \frac{1}{2}m_2V_2'^2 + \frac{1}{2}m_1V_1'^2$

$$m_1V_1^2 + 8m_1V_2^2 = 8m_1V_2'^2 + m_1V_1'^2$$

$$V_1^2 + 8V_2^2 = 8V_2'^2 + V_1'^2$$

$$9V_1^2 = 8V_2'^2 + (7V_1 - 8V_2')^2$$

$$72V_2'^2 - 112V_1V_2' + 40V_1^2 = 0$$

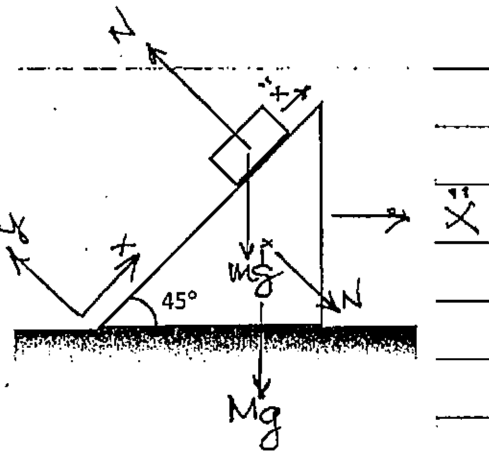
$$\Rightarrow \quad V_2' = \frac{5}{9}V_1 = \frac{5}{9}\sqrt{2g(h-2a)} \quad \text{or} \quad V_2' = V_1 \text{ (rejected)}$$

$$V_1' = \frac{23}{9}V_1 = \frac{23}{9}\sqrt{2g(h-2a)}$$

Maximum Height $= 5a + \frac{V_1'^2}{2g} = 5a + \frac{529}{81}(h-2a)$

8.

$$\begin{cases} M\ddot{X} = N\sin\theta & \text{—— (1)} \\ m(\ddot{x} + \ddot{X}\cos\theta) = -mg\sin\theta & \text{—— (2)} \\ -m\ddot{X}\sin\theta = N - mg\cos\theta & \text{—— (3)} \end{cases}$$



(3) $\Rightarrow$ $$-m\ddot{X}\frac{\sqrt{2}}{2} = N - mg\frac{\sqrt{2}}{2}$$

$$N = mg\frac{\sqrt{2}}{2} - m\ddot{X}\frac{\sqrt{2}}{2}$$

Sub. into (1),

$$M\ddot{X} = N\frac{\sqrt{2}}{2}$$

$$= \left(mg\frac{\sqrt{2}}{2} - m\ddot{X}\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2}$$

$$= \frac{mg}{2} - \frac{m\ddot{X}}{2}$$

$$\Rightarrow \ddot{X} = \frac{mg}{m + 2M}$$

HKPhO2016 MC9

Let a_n = downward acceleration of the nth pulley
Let T_n = tension of the string hanging down from the nth pulley

Consider the forces acting on the second pulley. Since it is massless,

$$T_2 = \frac{T_1}{2}$$

Similarly, we can prove $T_n = \frac{T_{n-1}}{2} = \frac{T_1}{2^{n-1}}$

Consider the forces acting on the first mass. Using Newton's law,

$$T_1 - mg = ma_1 \Rightarrow T_1 = mg + ma_2$$

Consider the forces acting on the second mass. Its acceleration in the lab frame is $a_3 - a_2$. Using Newton's law,

$$T_2 - \frac{m}{2}g = \frac{m}{2}(a_3 - a_2) \Rightarrow \frac{T_1}{2} = \frac{m}{2}(g + a_3 - a_2) \Rightarrow T_1 = m(g + a_3 - a_2)$$

In general, consider the forces acting on the kth mass. Its acceleration in the lab frame is $a_{k+1} - a_k$. Using Newton's law,

$$T_k - \frac{m}{2^{k-1}}g = \frac{m}{2^{k-1}}(a_{k+1} - a_k) \Rightarrow \frac{T_1}{2^{k-1}} = \frac{m}{2^{k-1}}(g + a_{k+1} - a_k) \Rightarrow T_1 = m(g + a_{k+1} - a_k)$$

This equation holds up to $k = n - 1$.

Consider the forces acting on the last pair of masses (that is, those hanging on the nth pulley). Their acceleration in the lab frame is a_n downward. Using Newton's law,

$$mg - T_n = ma_n \Rightarrow \frac{T_1}{2^{n-1}} = mg - ma_n$$

Adding the n equations above,

$$\left(n - 1 + \frac{1}{2^{n-1}}\right)T_1 = nmg + m(a_2 - 0) + m(a_3 - a_2) + \cdots + m(a_n - a_{n-1}) + m(0 - a_n) = nmg$$

Note that in summing the acceleration terms, the first term of each bracket cancels the second term of the following bracket.

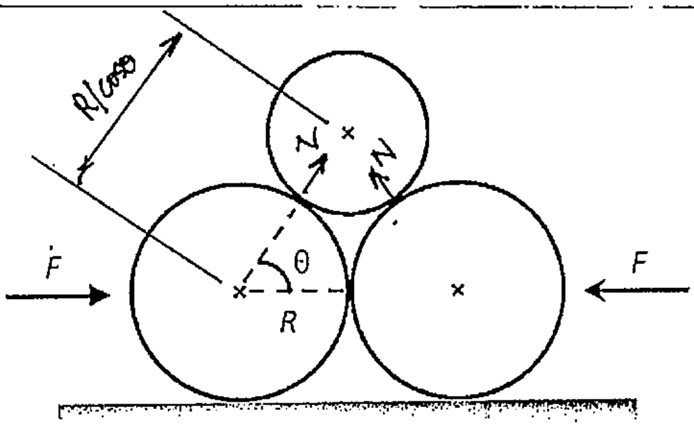
$$T_1 = \frac{nmg}{n - 1 + 2^{1-n}}$$

The acceleration of the mass hanging from the 1st pulley $= a_2 = \frac{T_1}{m} - g = \frac{1 - 2^{1-n}}{n - 1 + 2^{1-n}} g \to 0$

The acceleration of the lowest mass $= a_n = g - \frac{T_1}{m2^{n-1}} = \frac{(n-1)(1 - 2^{1-n})}{n - 1 + 2^{1-n}} g \to g$

Radius of the upper circle $= \dfrac{R}{\cos\theta} - R$

$= R\left(\dfrac{1}{\cos\theta} - 1\right)$



Mass of the upper circle

$= \sigma \pi R^2 \left(\dfrac{1}{\cos\theta} - 1\right)^2$

Keeping stationary :

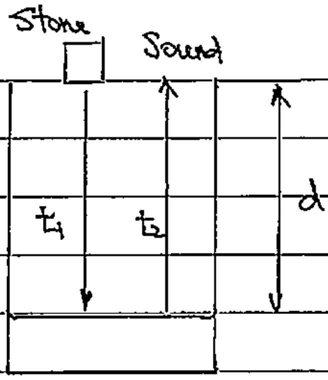
$$2N\sin\theta = \sigma\pi R^2\left(\frac{1}{\cos\theta} - 1\right)^2 g \cancel{\cos\theta}$$

$$N = \frac{\sigma\pi R^2}{2\sin\theta}\left(\frac{1}{\cos\theta} - 1\right)^2 g \cancel{\cos\theta}$$

$\cancel{F = N\sin}$

$$F = N\cos\theta = \frac{\sigma\pi R^2}{2\sin\theta}\,\frac{(1-\cos\theta)^2}{\cos(\theta)}\,g$$

11.



$$d = \frac{1}{2}(9.8)\,t_1^2 \Rightarrow d = 4.9\,t_1^2 \quad —(1)$$

$$t_1 + t_2 = 3s \Rightarrow t_2 = 3 - t_1 \quad —(2)$$

$$d = 340\,t_2 \quad —(3)$$

Sub. (2) into (3),

$$d = 340\,(3 - t_1)$$

$$= 1020 - 340\,t_1$$

$$= 1020 - 340\sqrt{\frac{d}{4.9}} \qquad \text{(from (1))}$$

Rearranging :

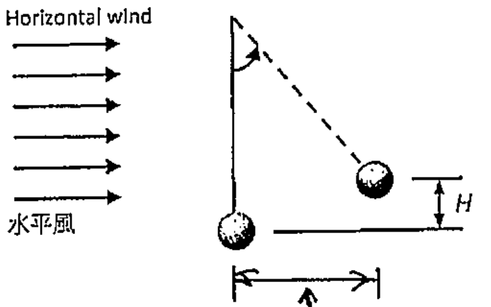
$$\cancel{d^2 - 25632d + 1020^2 = 0}$$

$$\cancel{d = -40.5\,m \quad \text{or} \quad d = 25.7\,m}$$

$$\cancel{\Rightarrow d = 25.7\,m}$$

$$d = 40.65\,m$$

$\sigma \, [N/m^2]$



$$\sqrt{L^2 - (L-H)^2} = \sqrt{2LH - H^2}$$

$$F = \sigma \times \pi r^2$$

Work Done by Wind $= \int F \cdot ds$

$$= F \times \sqrt{2LH - H^2}$$

$$= \sigma \pi r^2 \sqrt{2LH - H^2}$$

Energy Conservation :

$$\sigma \pi r^2 \sqrt{2LH - H^2} = mgH$$

$$\sigma^2 \pi^2 r^4 (2LH - H^2) = m^2 g^2 H^2$$

$$(m^2 g^2 + \sigma^2 \pi^2 r^4) H = 2L\sigma^2 \pi^2 r^4$$

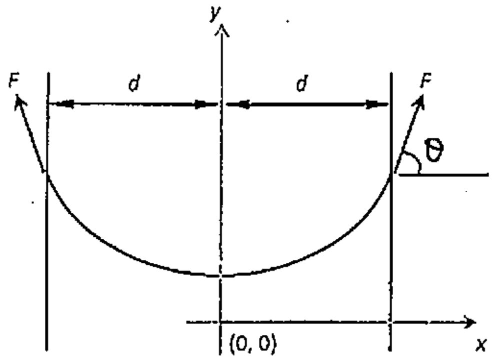
$$H = \frac{2L\sigma^2 \pi^2 r^4}{m^2 g^2 + \sigma^2 \pi^2 r^4}$$

13.

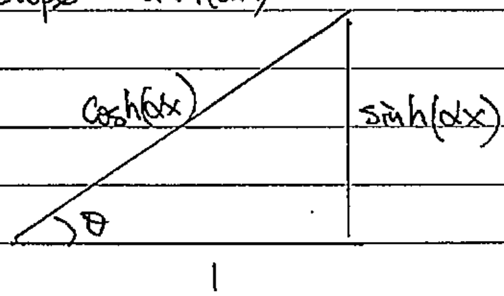
$$2F\sin\theta = mg$$

$$2F\tanh(\alpha d) = mg$$

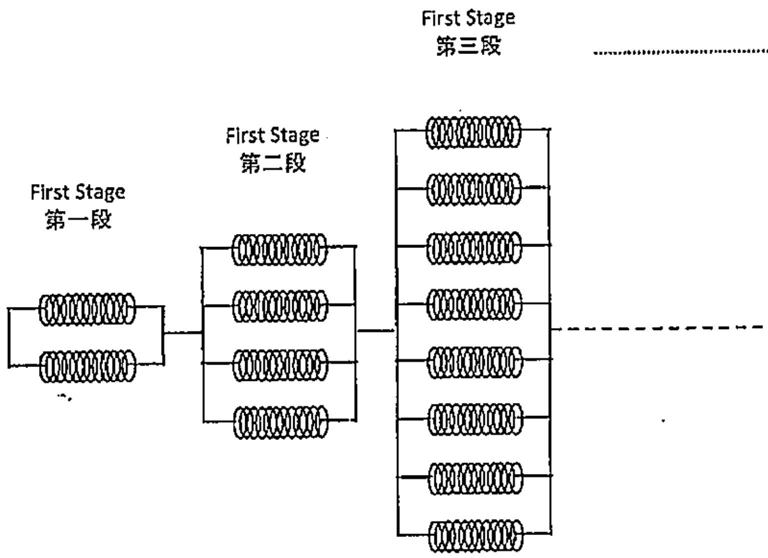
$$F = \frac{mg}{2\tanh(\alpha d)}$$



Slope = $\sinh(\alpha x)$



$$\Rightarrow \quad \sin\theta = \frac{\sinh(\alpha x)}{\cosh(\alpha x)} = \tanh(\alpha x)$$



Spring Constant	$2k$	$4k$	$8k$	......
Elongation	x	$\frac{x}{2}$	$\frac{x}{4}$	......

14. $$\frac{1}{k_{eq}} = \frac{1}{2k} + \frac{1}{4k} + \frac{1}{8k} + \cdots\cdots$$

$$= \frac{1}{k}\left(\frac{\frac{1}{2}}{1-\frac{1}{2}}\right)$$

$$k_{eq} = k$$

15. $$x + \frac{x}{2} + \frac{x}{4} + \cdots\cdots = 10\text{cm}$$

$$x\left(\frac{1}{1-\frac{1}{2}}\right) = 10\text{cm}$$

$$\Rightarrow \quad x = 5\text{cm}$$

Fourth Stage Elongation $= \frac{5\text{cm}}{8} = 0.625\text{ cm}$

16.

Trajectory Equation:

$$y = (\tan\theta_1)(x) - \frac{g}{2V_1^2\cos^2\theta_1}(x^2).$$

$$\begin{cases} x = d\cos\phi \\ y = d\sin\phi \end{cases}$$

$$\Rightarrow \quad d\sin\phi = (\tan\theta_1)(d\cos\phi) - \frac{g}{2V_1^2\cos^2\theta_1}(d^2\cos^2\phi)$$

$$\sin\phi\cos^2\theta_1 = \sin\theta_1\cos\theta_1\cos\phi - \frac{gd\cos^2\phi}{2V_1^2}$$

$$d_1 = \frac{2V_1^2}{g\cos^2\phi}\cos\theta_1\left(\sin\theta_1\cos\phi - \sin\phi\cos\theta_1\right)$$

$$= \frac{2V_1^2}{g\cos^2\phi}\cos\theta_1\sin(\theta_1 - \phi)$$

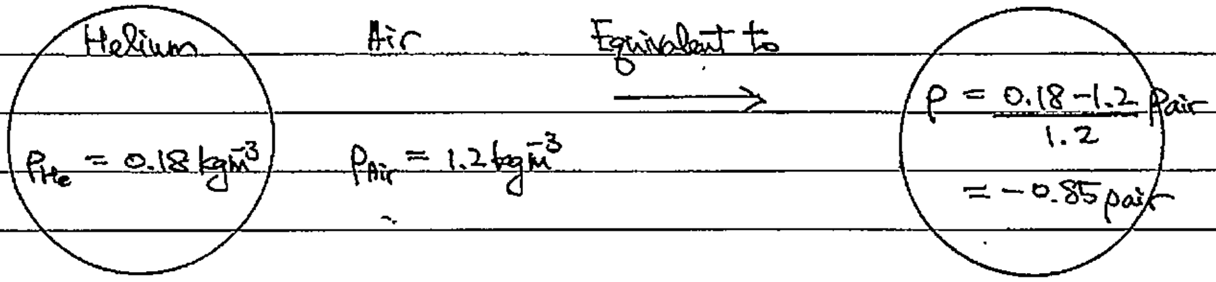
Similarly, $\quad d_2 = \dfrac{2V_2^2}{g\cos^2\phi}\cos\theta_2\sin(\theta_2 - \phi)$

$d_1 = d_2, \quad \Rightarrow \quad 2V_1^2\cos\theta_1\sin(\theta_1 - \phi) = 2V_2^2\cos\theta_2\sin(\theta_2 - \phi)$

$$\left(\frac{V_1}{V_2}\right)^2 = \frac{\cos\theta_2\sin(\theta_2 - \phi)}{\cos\theta_1\sin(\theta_1 - \phi)}$$

17.

Consider an equivalent case that the density of the balloon is $-0.85\rho_{air}$ (Density of air), and neglect the effect of air.



Effective gravitational acceleration $= -0.85\,g$

Effective centrifugal acceleration $= -0.85\,a$

$$a = \frac{v^2}{R} = \frac{(105\,km/hour)^2}{1000\,m} = 0.85\,ms^{-2}$$

$$\Rightarrow \tan\theta = \frac{-0.85}{-0.85}\left(\frac{a}{g}\right)$$

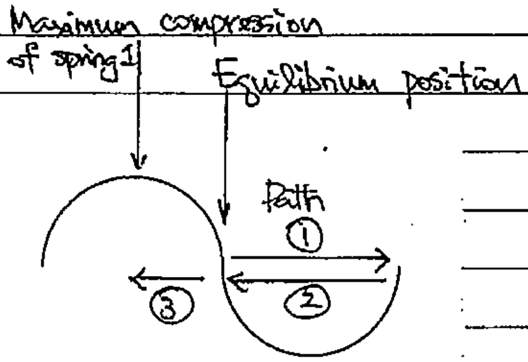
$$= \frac{0.85}{9.8}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{0.85}{9.8}\right)$$

$$= 5^\circ$$

18.

Effective spring constant, $k = k_1 + k_2 = 9$ N/m.



Maximum compression of Spring 1 occurs after $\frac{3}{4}$ period,

$\Rightarrow$ The time that the block will take

$$= \frac{3T}{4}$$

$$= \frac{3}{4}(2\pi)\sqrt{\frac{M}{k}}$$

$$= \frac{3}{4}(2\pi)\sqrt{\frac{1}{9}}$$

$$= \frac{\pi}{2} \text{ seconds}$$

19.

$$\text{Center of Mass} = \frac{\sum m_i x_i}{M}$$

$$\begin{aligned}\sum m_i x_i &= \pi r^2 (r) + \pi\left(\frac{r}{2}\right)^2\left(2r + \frac{r}{2}\right) \\ &\quad + \pi\left(\frac{r}{4}\right)^2\left[2r + 2\left(\frac{r}{2}\right) + \frac{r}{4}\right] \\ &\quad + \pi\left(\frac{r}{8}\right)^2\left[2r + 2\left(\frac{r}{2}\right) + 2\left(\frac{r}{4}\right) + \frac{r}{8}\right] \\ &\quad + \pi\left(\frac{r}{16}\right)^2\left[2r + 2\left(\frac{r}{2}\right) + 2\left(\frac{r}{4}\right) + 2\left(\frac{r}{8}\right) + \frac{r}{16}\right] \\ &\quad + \cdots\cdots \\ &= \pi r^3\left[1 + \frac{5}{8} + \frac{13}{64} + \frac{29}{512} + \frac{61}{4096} + \cdots\cdots\right] \\ &= \pi r^3 \sum_{n=1}^{\infty} 8^{1-n}\left(2^{n+1} - 3\right) \\ &= \frac{40}{21}\pi r^3\end{aligned}$$

$$\begin{aligned}M &= \pi r^2 + \pi\left(\frac{r}{2}\right)^2 + \pi\left(\frac{r}{4}\right)^2 + \pi\left(\frac{r}{8}\right)^2 + \cdots\cdots \\ &= \pi r^2\left(1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots\cdots\right) \\ &= \pi r^2\left(1 + \sum_{n=1}^{\infty} 4^{-n}\right) \\ &= \pi r^2\left(1 + \frac{1}{3}\right) \\ &= \pi r^2\left(\frac{4}{3}\right)\end{aligned}$$

Note: The mass proportional constant is omitted. It will be cancelled finally.

$$\Rightarrow \text{Center of Mass} = \frac{\frac{40}{21}\pi r^3}{\frac{4}{3}\pi r^2} = \frac{10}{7} r$$

20.

$$\begin{cases} mv = mv_{1f} + 2mv_{45f} \\ \frac{1}{2}mv^2 = \frac{1}{2}mv_{1f}^2 + \frac{1}{2}(2m)v_{45f}^2 \end{cases}$$

Solutions give :

$$\begin{cases} v_{1f} = -\frac{1}{3}v \\ v_{45f} = \frac{2}{3}v \end{cases}$$

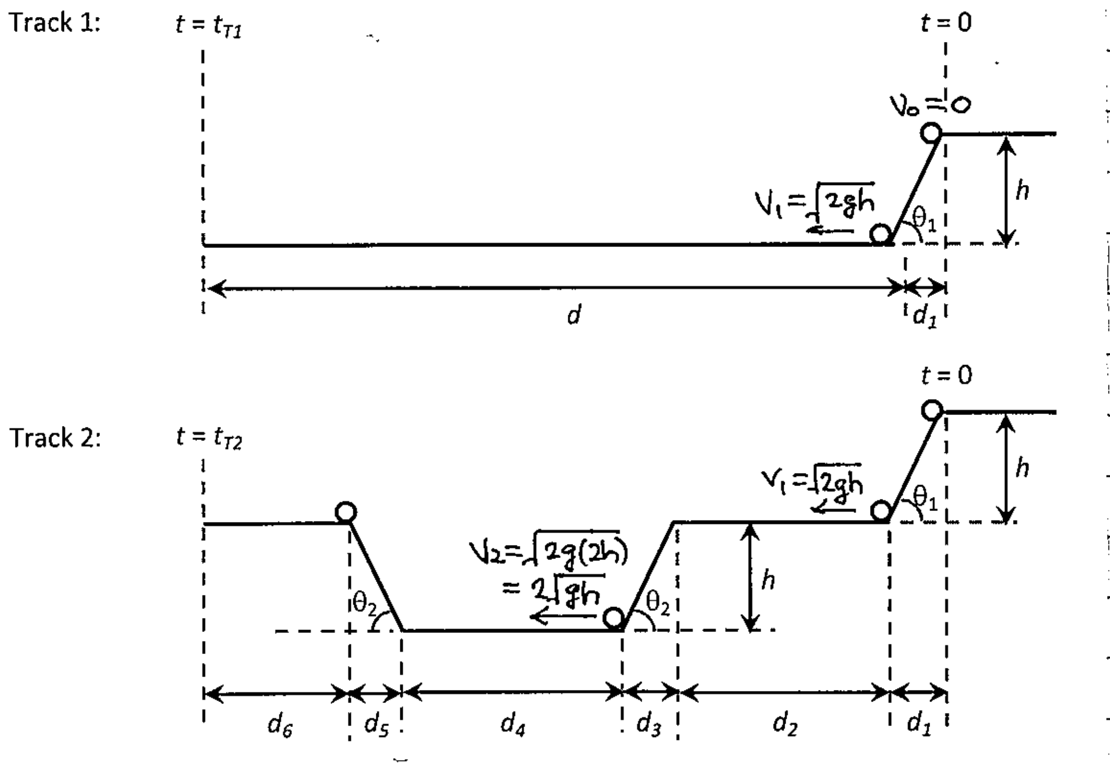
1.

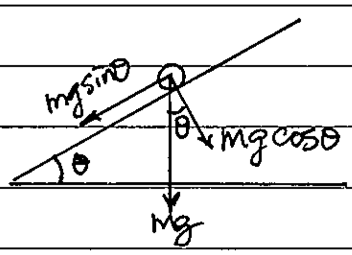
(a) $d = d_2 + d_3 + d_4 + d_5 + d_6$

$$= d_2 + \frac{h}{\sin\theta_2} + d_4 + \frac{h}{\sin\theta_2} + d_6$$

$$= d_2 + d_4 + d_6 + \frac{2h}{\sin\theta_2}$$

(b)





$$t_{d_1} = \frac{\sqrt{2gh}}{g\sin\theta_1}$$

$$t_{d_3} = \frac{\sqrt{gh}\,(2-\sqrt{2})}{g\sin\theta_2}$$

$$t_{d_4} = \frac{d_4}{2\sqrt{gh}}$$

$$t_d = \frac{d}{\sqrt{2gh}}$$

$$t_{d_5} = t_{d_3} = \frac{\sqrt{gh}\,(2-\sqrt{2})}{g\sin\theta_2}$$

$$t_{d_2} = \frac{d_2}{\sqrt{2gh}}$$

$$t_{d_6} = \frac{d_6}{\sqrt{2gh}}$$

1.

(b) $t_{T_1} = t_{d_1} + t_d$

$$= \frac{\sqrt{2gh}}{g\sin\theta_1} + \frac{d}{\sqrt{2gh}}$$

$$t_{T_2} = t_{d_1} + t_{d_2} + \cdots + t_{d_6}$$

$$= \frac{\sqrt{2gh}}{g\sin\theta_1} + \frac{d_2}{\sqrt{2gh}} + \frac{\sqrt{gh}\,(2-\sqrt{2})}{g\sin\theta_2} + \frac{d_4}{2\sqrt{gh}} + \frac{\sqrt{gh}\,(2-\sqrt{2})}{g\sin\theta_2} + \frac{d_6}{\sqrt{2gh}}$$

$$= \frac{\sqrt{2gh}}{g\sin\theta_1} + \frac{d_2+d_6}{\sqrt{2gh}} + \frac{d_4}{2\sqrt{gh}} + \frac{2\sqrt{gh}\,(2-\sqrt{2})}{g\sin\theta_2}$$

(c) $$\Delta t = \frac{d}{\sqrt{2gh}} - \frac{d_2+d_6}{\sqrt{2gh}} - \frac{d_4}{2\sqrt{gh}} - \frac{2\sqrt{gh}\,(2-\sqrt{2})}{g\sin\theta_2}$$

$$= \frac{1}{\sqrt{2gh}}\left(d - d_2 - d_4 + d_4 - \frac{d_4}{\sqrt{2}} - d_6\right) - \frac{2\sqrt{gh}\,(2-\sqrt{2})}{g\sin\theta_2}$$

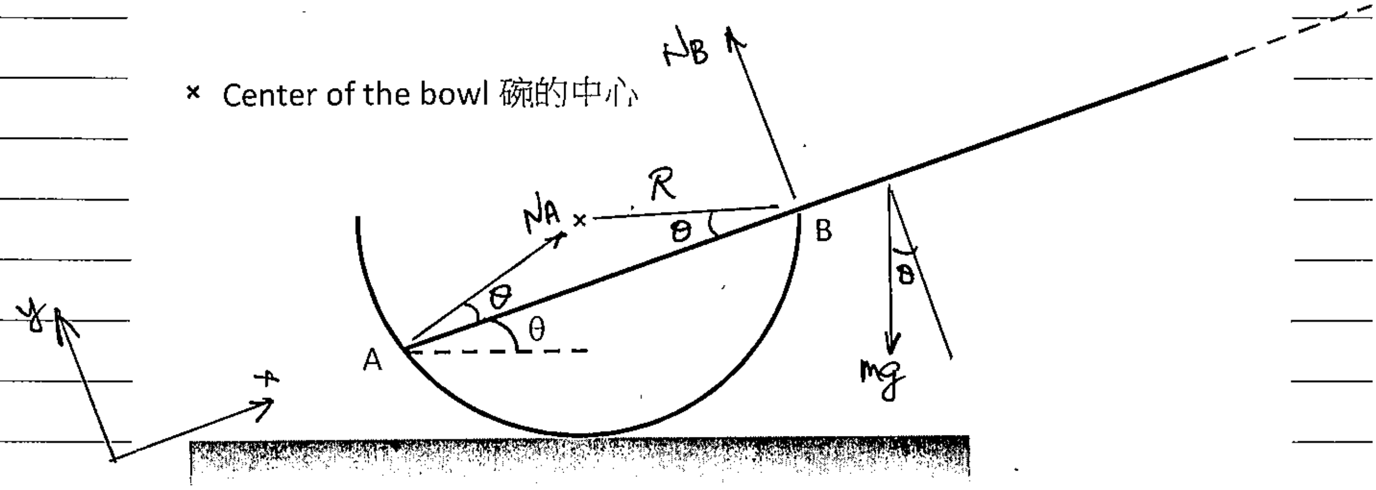
$$= \frac{1}{\sqrt{2gh}}\left(\frac{2h}{\sin\theta_2} + \frac{\sqrt{2}-1}{\sqrt{2}}\,d_4\right) - \frac{2\sqrt{gh}\,(2-\sqrt{2})}{g\sin\theta_2}$$

$$= \frac{\sqrt{2h}}{\sqrt{g}\sin\theta_2} - \frac{2\sqrt{h}\,(2-\sqrt{2})}{\sqrt{g}\,\sin\theta_2} + \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)\frac{1}{\sqrt{2gh}}\,d_4$$

$$= \frac{\sqrt{h}\,(3\sqrt{2}-4)}{\sqrt{g}\,\sin\theta_2} + \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)\frac{d_4}{\sqrt{2gh}}$$

2.

(a)



$$\Sigma F_x = 0, \quad N_A \cos\theta = mg\sin\theta \quad \text{——— (1)}$$

$$\Sigma F_y = 0, \quad N_A \sin\theta + N_B = mg\cos\theta \quad \text{——— (2)}$$

$\Sigma M_A = 0$, Moment about A :

Length of AB

$$N_B(2R\cos\theta) - \frac{L}{2}mg\cos\theta = 0$$

$$\Rightarrow N_B = \frac{Lmg}{4R}$$

Sub. into (2),

$$N_A \sin\theta + \frac{Lmg}{4R} = mg\cos\theta$$

$$N_A = \left(mg\cos\theta - \frac{Lmg}{4R}\right)\frac{1}{\sin\theta} \quad \text{——— (3)}$$

Sub. into (1),

$$\left(mg\cos\theta - \frac{Lmg}{4R}\right)\frac{1}{\sin\theta}\cos\theta = mg\sin\theta$$

$$\frac{4R\cos^2\theta - L\cos\theta}{4R} = \sin^2\theta$$

2.

(a) $4R\cos^2\theta - L\cos\theta = 4R(1-\cos^2\theta)$

$$8R\cos^2\theta - L\cos\theta - 4R = 0$$

$$\cos\theta = \frac{L \pm \sqrt{L^2 + 128R^2}}{16R}$$

$$\Rightarrow \cos\theta = \frac{L + \sqrt{L^2 + 128R^2}}{16R}$$

$$AB = 2R\left(\frac{L + \sqrt{L^2 + 128R^2}}{16R}\right)$$

(b) When $L = 4R$, $\cos\theta = 1$

$$\Rightarrow \theta = 0^\circ$$

3.

(a) Consider the upper sphere, energy conservation gives

$$\frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\dot{\beta}^2 + mg(R+r)\cos\alpha = mg(R+r)$$

$$v = \dot{\beta}r = (R+r)\dot{\alpha}$$

$$\dot{\alpha}^2 = \dot{\beta}\,\frac{r^2}{(R+r)^2}$$

$$\frac{v^2}{2} + \frac{1}{5}r^2\dot{\beta}^2 + g(R+r)\cos\alpha = g(R+r)$$

$$\frac{\dot{\beta}^2 r^2}{2} + \frac{r^2\dot{\beta}^2}{5} + g(R+r)\cos\alpha = g(R+r)$$

$$\frac{7r^2\dot{\beta}^2}{10} = g(R+r)(1-\cos\alpha)$$

$$r^2\dot{\beta}^2 = \frac{10}{7}g(R+r)(1-\cos\alpha)$$

$$\dot{\beta}^2 = \frac{10}{7r^2}g(R+r)(1-\cos\alpha)$$

$$\dot{\alpha}^2 = \frac{10}{7}g\,\frac{(1-\cos\alpha)}{R+r}$$

$$v = (R+r)\dot{\alpha}$$

$$= (R+r)\sqrt{\frac{10}{7}g\,\frac{(1-\cos\alpha)}{R+r}}$$

$$= \sqrt{\frac{10}{7}g(R+r)(1-\cos\alpha)}$$

3.

(b) $$mg\cos\alpha - N = \frac{mv^2}{R+r}$$

Upper sphere will detach from the lower sphere when $N=0$.

$$g\cos\alpha = \frac{v^2}{R+r}$$

$$g\cos\alpha = \frac{10}{7}g\,(R+r)(1-\cos\alpha)\times\frac{1}{R+r}$$

$$\cos\alpha = \frac{10}{7}(1-\cos\alpha)$$

$$\Rightarrow \cos\alpha = \frac{10}{17}$$

$$\alpha \approx 54^\circ$$

(c) $$\begin{cases} mg\sin\alpha - f = m\dot{v} \quad \Rightarrow \quad \dot{v} = g\sin\alpha - \dfrac{f}{m} \\ fr = \left(\dfrac{2}{5}mr^2\right)\ddot{\beta} \\ v = \dot{\beta}r = (R+r)\dot{\alpha} \quad \Rightarrow \quad \dot{v} = \ddot{\beta}r = (R+r)\ddot{\alpha} \end{cases}$$

$$fr = \left(\frac{2}{5}mr^2\right)\ddot{\beta}$$

$$= \left(\frac{2}{5}mr^2\right)\frac{\dot{v}}{r}$$

$$= \left(\frac{2}{5}mr^2\right)\frac{1}{r}\left(g\sin\alpha - \frac{f}{m}\right)$$

$$f = \left(\frac{2}{5}m\right)\left(g\sin\alpha - \frac{f}{m}\right)$$

$$= \frac{2}{5}mg\sin\alpha - \frac{2}{5}f$$

3.

(c) $\Rightarrow \quad f = \frac{2}{7} mg \sin\alpha \quad$ (Given)

Condition to slide: $\quad f = \mu N$

$$\Rightarrow \quad \frac{2}{7} mg \sin\alpha = \mu N$$

$$\frac{2}{7} mg \sin\alpha = \mu\left(mg\cos\alpha - \frac{mv^2}{R+r} \right)$$

Substitute v using the results from (a),

$$\frac{2}{7} g \sin\alpha = \mu\left[g\cos\alpha - \frac{10}{7} g (1-\cos\alpha) \right]$$

$$\frac{2}{7} \sin\alpha = \mu\cos\alpha - \frac{10}{7}\mu + \frac{10}{7}\mu\cos\alpha$$

$$2\sin\alpha = 17\mu\cos\alpha - 10\mu$$

$$2\left(\sqrt{1-\cos^2\alpha}\right) = 17\mu\cos\alpha - 10\mu$$

$$4(1-\cos^2\alpha) = 289\mu^2\cos^2\alpha + 100\mu^2 - 340\mu^2\cos^2\alpha$$

$$\cos^2\alpha\,(289\mu^2+4) - \cos\alpha\,(340\mu^2) + 100\mu^2 - 4 = 0$$

$$\cos\alpha = \frac{340\mu^2 \pm \sqrt{340^2\mu^4 - 4(289\mu^2+4)(100\mu^2-4)}}{2(289\mu^2+4)}$$

$$= \frac{340\mu^2 \pm \sqrt{16(189\mu^2+4)}}{2(289\mu^2+4)}$$

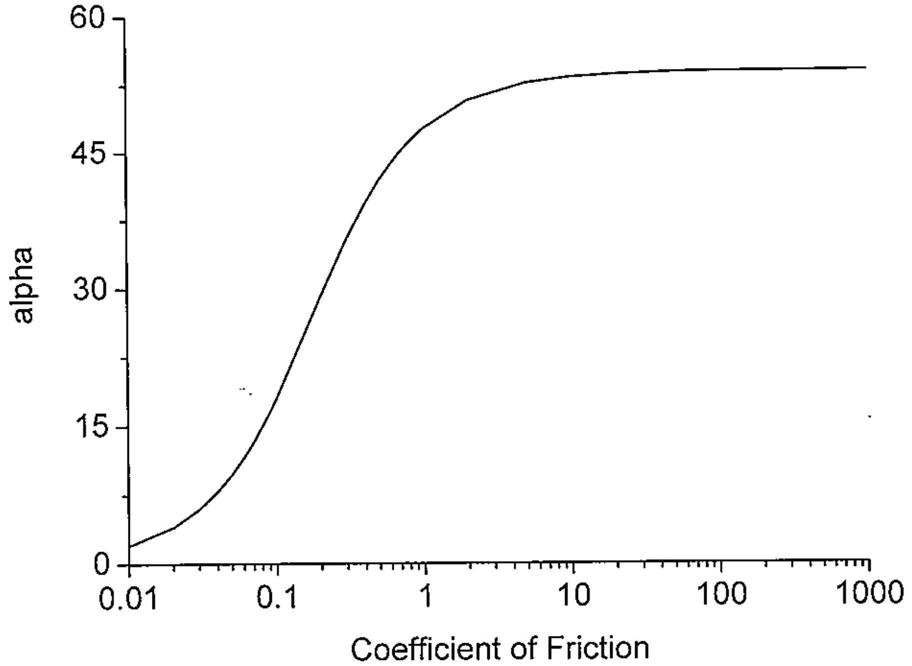
$$= \frac{170\mu^2 \pm 2\sqrt{189\mu^2+4}}{289\mu^2+4}$$

Since $0^\circ \leq \alpha \leq 90^\circ$,

$$\Rightarrow \quad \cos\alpha = \frac{170\mu^2 + 2\sqrt{189\mu^2+4}}{289\mu^2+4}$$

3.

(c) For reference :



(d) When $\mu = 0$, $\cos\alpha = 1$ and $\alpha = 0°$

Remark: without friction, the upper sphere slides immediately.

When $\mu \to \infty$, $\cos\alpha = \dfrac{170 + 2\sqrt{189/\mu^2 + 4/\mu^4}}{289 + 4/\mu^4}$

$$= \frac{170}{289}$$

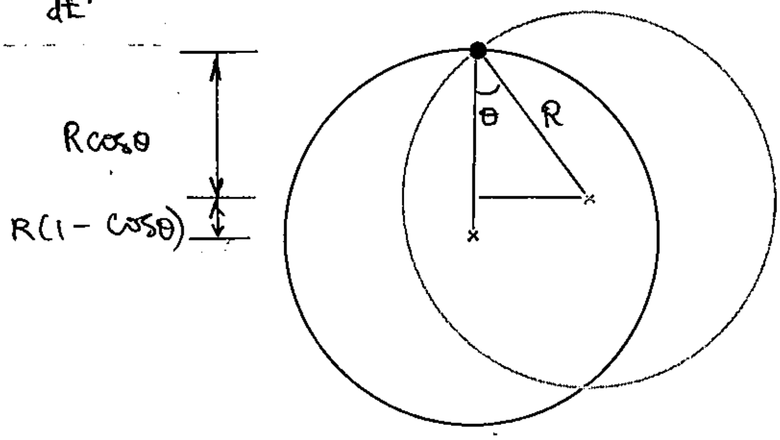
$$= \frac{10}{17}$$

$$\Rightarrow \alpha \approx 54°$$

4.

$$KE = \frac{1}{2}(2M_1R_1^2)\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}(2M_2R_2^2)\left(\frac{d\theta}{dt}\right)^2$$

$$= (M_1R_1^2 + M_2R_2^2)\left(\frac{d\theta}{dt}\right)^2$$



(a)

$$PE = -M_1 g R_1(1-\cos\theta) + M_2 g R_2 (1-\cos\theta)$$

(b) $KE + PE = \text{constant}$

$$(M_1R_1^2 + M_2R_2^2)\left(\frac{d\theta}{dt}\right)^2 - M_1 g R_1(1-\cos\theta) + M_2 g R_2(1-\cos\theta) = \text{constant}$$

$$(M_1R_1^2 + M_2R_2^2)\left(\frac{d\theta}{dt}\right)^2 - M_1 g R_1\left(2\sin^2\frac{\theta}{2}\right) + M_2 g R_2\left(2\sin^2\frac{\theta}{2}\right) = \text{constant}$$

For small θ, $\sin^2\left(\frac{\theta}{2}\right) \approx \frac{\theta^2}{4}$.

$$(M_1R_1^2 + M_2R_2^2)\left(\frac{d\theta}{dt}\right)^2 + \frac{\theta^2}{2} g (M_2R_2 - M_1R_1) = \text{constant}$$

$$\frac{1}{2}(2)(M_1R_1^2 + M_2R_2^2)\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2} g (M_2R_2 - M_1R_1)\theta^2 = \text{constant}$$

(c) $k_{eff} = g(M_2R_2 - M_1R_1)$

$$M_{eff} = 2(M_1R_1^2 + M_2R_2^2)$$

$$\omega = \sqrt{\frac{k_{eff}}{M_{eff}}} = \sqrt{\frac{g(M_2R_2 - M_1R_1)}{2(M_1R_1^2 + M_2R_2^2)}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2(M_1R_1^2 + M_2R_2^2)}{g(M_2R_2 - M_1R_1)}}$$

Alternative Method to (b) and (c):

$$(m_1R_1^2 + m_2R_2^2)\left(\frac{d\theta}{dt}\right)^2 - m_1gR_1(1-\cos\theta) + M_2gR_2(1-\cos\theta) = \text{constant}$$

$$(m_1R_1^2 + M_2R_2^2)(2)\left(\frac{d\theta}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right) - M_1gR_1\sin\theta\,\frac{d\theta}{dt} + M_2gR_2\sin\theta\,\frac{d\theta}{dt} = 0$$

$$2(m_1R_1^2 + m_2R_2^2)\left(\frac{d^2\theta}{dt^2}\right) + g(M_2R_2 - m_1R_1)\sin\theta = 0$$

$$\frac{d^2\theta}{dt^2} \approx -\frac{g(M_2R_2 - M_1R_1)\,\theta}{2(M_1R_1^2 + M_2R_2^2)}$$

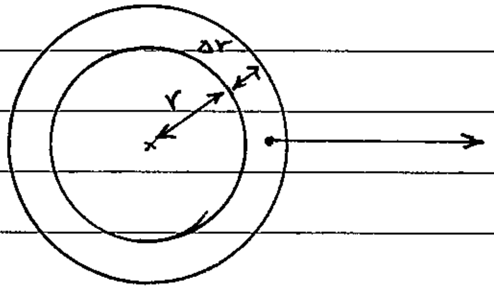
$$\Rightarrow\ \omega = \frac{g(M_2R_2 - M_1R_1)}{2(M_1R_1^2 + m_2R_2^2)}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2(m_1R_1^2 + m_2R_2^2)}{g(M_2R_2 - m_1R_1)}}$$

(d) When $m_1R_1 \to 0$,

$$T = 2\pi\sqrt{\frac{2m_2R_2^2}{g\,m_2R_2}} = 2\pi\sqrt{\frac{2R_2}{g}}$$

5.



Remove this thin shell to infinity

Energy to remove a thin shell at radius r :

$$E = -\frac{G M_r m_r}{r}$$

where M_r = mass within the radius $r = \frac{4}{3}\pi r^3 \rho$

m_r = Mass of thin shell at radius r and thickness Δr.
$= -4\pi r^2 \rho \Delta r$ (ρ = Density of the Sun)

Energy to remove the ith thin shell :

$$E_i = -\frac{G}{r}\left(\frac{4\pi}{3} r_i^3 \rho\right)\left(-4\pi r_i^2 \rho \Delta r\right)$$

$$= G \frac{(4\pi)^2}{3} \rho^2 r_i^4 \Delta r$$

$$E_{total} = \sum_i E_i$$

$$= \frac{(4\pi)^2}{3} G \rho^2 \sum_i \left(\frac{R_s}{N} i\right)^4 \left(\frac{R_s}{N}\right)$$

[Suppose the Sun is divided into N thin shells,
where $N \to \infty$]

5.

$$E_{Total} = \frac{(4\pi)^2}{3} \frac{G\rho^2}{N^5} R_S^5 \sum_i i^4$$

$$= \frac{(4\pi)^2}{3} \frac{G\rho^2}{N^5} R_S^5 \left(\frac{6N^5}{30} \right)$$

$$= \frac{(4\pi)^2}{3} G\rho^2 \frac{R_S^5}{5}$$

$$= \frac{(4\pi)^2}{3} G \left(\frac{M_S}{4/3\, \pi R_S^3} \right)^2 \frac{R_S^5}{5}$$

$$= \frac{3}{5} G \frac{M_S^2}{R_S}$$