# Hong Kong Physics Olympiad 2018
# 2018 年香港物理奧林匹克競賽

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**香港物理學會**

**Hong Kong Physics Olympiad Committee**

**香港物理奧林匹克委員會**

**13 May, 2018**

**2018 年 5 月 13 日**

# Rules and Regulations 競賽規則

1. All questions are in bilingual versions. You can answer in either Chinese or English, but only ONE language should be used throughout the whole paper.

   所有題目均為中英對照。你可選擇以中文或英文作答，惟全卷必須以單一語言作答。

2. The multiple-choice answer sheet will be collected 1.5 hours after the start of the contest. You can start answering the open-ended questions any time after you have completed the multiple-choice questions without waiting for announcements.

   選擇題的答題紙將於比賽後一小時三十分收回。若你在這之前已完成了選擇題，你亦可開始作答開放式題目，而無須等候任何宣佈。

3. On the cover of the answer book and the multiple-choice answer sheet, please write your 8-digit Contestant Number, English Name, and Seat Number.

   在答題簿封面及選擇題答題紙上，請填上你的 8 位數字參賽者號碼、英文姓名、及座位號碼。

4. After you have made the choice in answering a multiple choice question, fill the corresponding circle on the multiple-choice answer sheet **fully** using a HB pencil.

   選定選擇題的答案後，請將選擇題答題紙上相應的圓圈用 HB 鉛筆**完全塗黑**。

5. The open-ended problems are quite long. Please read the whole problem first before attempting to solve them. If there are parts that you cannot solve, you are allowed to treat the answer as a known answer to solve the other parts.

   開放式問答題較長，請將整題閱讀完後再著手解題。若某些部分不會做，也可把它們的答案當作已知來做其他部分。

**The following symbols and constants are used throughout the examination paper unless otherwise specified:**
**除非特別注明，否則本卷將使用下列符號和常數 ：**

| | | |
|---|---|---|
| Gravitational acceleration on Earth surface<br>地球表面重力加速度 | $g$ | 9.8 m $s^{-2}$ |
| Gravitational constant<br>重力常數 | $G$ | $6.67 \times 10^{-11}$ $m^3 kg^{-1} s^{-2}$ |

**Trigonometric identities:**

**三角學恆等式 ：**

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2}$$

$$\cos x - \cos y = 2 \sin \frac{x + y}{2} \sin \frac{y - x}{2}$$

$$\sin x \sin y = \frac{1}{2} \cos \frac{x - y}{2} - \frac{1}{2} \cos \frac{x + y}{2}$$

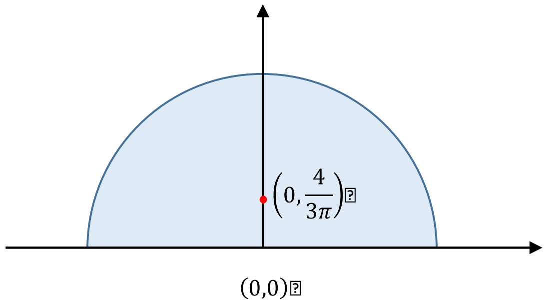
$$\cos x \cos y = \frac{1}{2} \cos \frac{x - y}{2} + \frac{1}{2} \cos \frac{x + y}{2}$$

$$\sin x \cos y = \frac{1}{2} \sin \frac{x + y}{2} + \frac{1}{2} \sin \frac{x - y}{2}$$

**Multiple Choice Questions (2 Marks Each) 選擇題(每題 2 分)**

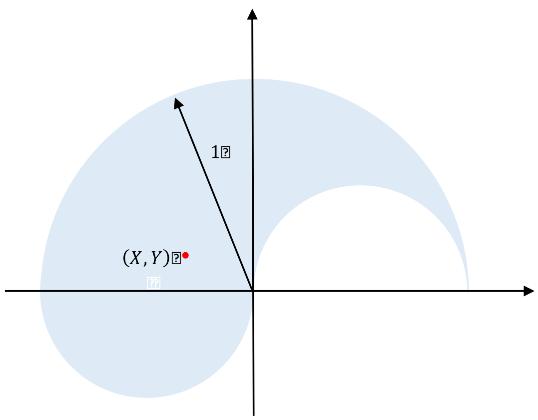
1. It is given that the center of mass of a uniform semi-disc with unit radius is located as shown in the figure below. Find the center of mass of the following figure.

1. 已知單位半徑的均勻半圓盤的質心位置如下圖所示。




Find the center of mass of the following figure.

找到下圖的質心位置。




A. $(X, Y) = \left(-\frac{1}{4}, \frac{2}{\pi}\right)$
B. $(X, Y) = \left(-\frac{1}{4}, \frac{1}{\pi}\right)$
C. $(X, Y) = \left(-\frac{1}{3}, \frac{1}{2\pi}\right)$
D. $(X, Y) = \left(-\frac{1}{2}, \frac{1}{2\pi}\right)$
E. $(X, Y) = \left(-\frac{1}{3}, \frac{1}{\pi}\right)$

Solution:
We can decompose the system into 3 hemisphere where one of them has negative mass density.

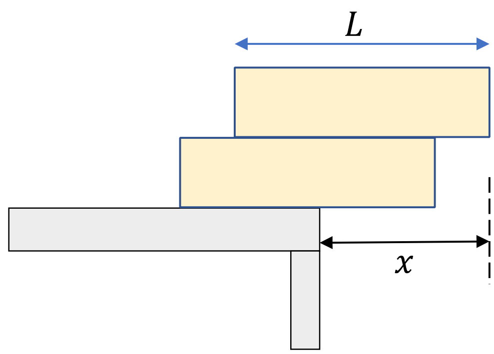
As the radius is doubled, the mass is 4 times bigger.

$$x_{cm} = \frac{1(-1/2) + (-1)(1/2) + 4(0)}{1 - 1 + 4} = -\frac{1}{4}$$

$$y = \frac{1(-\frac{2}{3\pi}) + (-1)(\frac{2}{3\pi}) + 4(\frac{4}{3\pi})}{1 - 1 + 4} = \frac{1}{3\pi}\left(\frac{12}{4}\right) = \frac{1}{\pi}$$

2. Two identical blocks with uniform mass and length $L$ are stacked at the table edge without falling off the table. What is the maximum overhang distance $x$ of two blocks?

2. 桌子邊緣堆放兩塊質量均勻，長度為 $L$ 的相同物塊，物塊不會從桌子上掉下來。兩個物塊的最大懸垂距離 $x$ 是多少？




A. $\frac{L}{4}$ B. $\frac{L}{3}$ C. $\frac{L}{2}$ D. $\frac{3L}{4}$ E. $L$

Solution:

The center of gravity of the top block can be as far out as the edge of the lower block. The center of gravity of the combined blocks is then 3L/4 to the left of the right edge of the upper block, so the maximum overhang is 3L/4.

3. A 50 kg woman stands up in a 75 kg boat of 10 m long. She walks from one end of the boat to the other end. If you ignore resistance to motion of boat in the water and assume that the mass of the boat is uniformly distributed, how far does the boat move during this process?

3. 一位 50 kg 的婦人站在一艘 10 m 長、75 kg 的船上。她從船的一端走到另一端。如果能忽略船在水上運動的阻力，並假設船的質量是均勻分佈的，在這個過程中船會移動多遠？

A. 2.0 m B. 4.0 m C. 5.0 m D. 6.7 m E. 8.0 m

Solution:

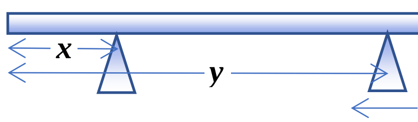
Let the center of mass of the system be located at the origin. Then the center of mass of the boat is located at 2 m to, say, the left side of the origin, and the woman is located at 3 m to the right side of the origin.

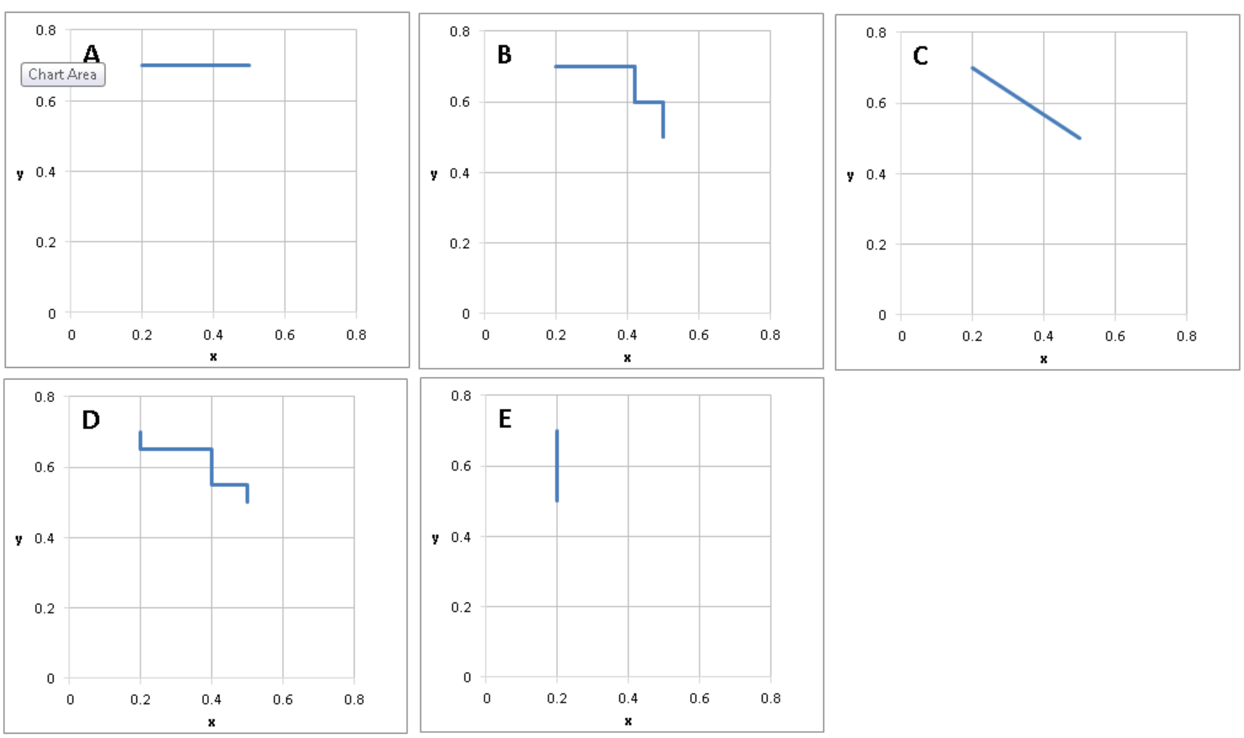
After the woman walks to the other end of the boat, the center of mass of the boat becomes located at 2 m to the right side of the origin, and the woman at 3 m to the left side of the origin.

Hence the displacement of the boat is 4 m.

4. A uniform rod of length 1 m is supported at two points each by a pivot. The left pivot is fixed and the right one is movable, and the coefficients of friction between the pivots and the rod are the same. The initial positions of the two pivots are $x$ = 0.2 m and $y$ = 0.7 m from the left end respectively. Now a leftward external force is horizontally applied to the right pivot. This external force is gradually increased from 0 until sliding motion takes place. Which of the following figures is a possible trajectory of $x$ and $y$?

4. 一根長度為1 m 的均勻桿通過支點分別支撐在兩個點上。左側支點是固定的，右側的則可移動，兩個支點間的摩擦系數相等。兩個支點的初始位置分別在左端的 $x = 0.2$ m 和 $y = 0.7$ m 現有一向左外力水平地施加在右側支點。這個外力從 0 開始逐漸增加，直到發生滑動。下列哪個圖是 $x$ 和 $y$ 的可能軌跡？

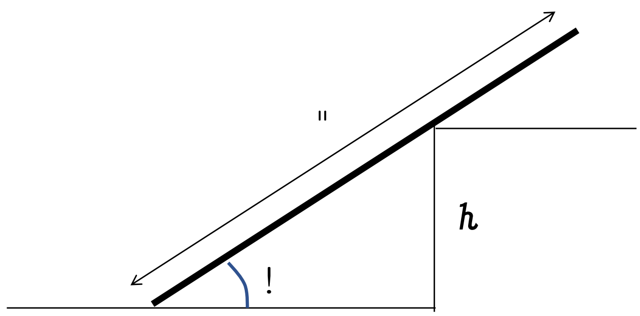



Answer: B. Since the left pivot is further from the center of mass, its normal reaction is smaller. Hence the external force on the left pivot is equal to the limiting friction and it will slide. On the other hand, the external force on the right pivot is less than the limiting friction and it will not slide. As the sliding continues, the limiting friction of the right pivot decreases and it starts to slide whereas the left pivot becomes stationary. The alternating movement continues until the two pivots meet at the center of mass.

5. A uniform ladder of mass $M = 10$ kg and length $L = 1$ m is leaning at an angle $\theta = 30°$ against a step whose height is $h = 0.5$ m. There is static friction between the ladder and the ground but negligible friction between the ladder and the step. What is the minimum value of the coefficient of static friction between the ladder and ground that would be necessary to keep the ladder from moving?

5. 質量 $M = 10$ kg 和長度 $L = 1$ m 的勻質梯子斜靠在高度為 $h = 0.5$ m 的梯級上，與地面形成一個 $\theta = 30°$ 的夾角。梯子和地面之間存在靜摩擦，但梯子梯級和之間的摩擦可以忽略不計。為防止梯子移動，梯子和地面之間的靜摩擦係數的最小值是多少？




A. 0.15 B. 0.25 C. 0.35 D. 0.45 E. 0.55

Solution:
Let $N_1$ be the upward normal force of the ground and $N_2$ be the normal force from the corner of the step.
Vertical force:

$$N_1 + N_2 \cos\theta - Mg = 0$$

Horizontal force:

$$-N_2 \sin\theta + f = 0$$

Torque relative to the ground,

$$-Mg\frac{L}{2}\cos\theta + N_2\frac{h}{\sin\theta} = 0$$

We get

$$f = \frac{MgL\sin^2\theta\cos\theta}{2h} \quad and \quad N_1 = \frac{Mg(2h - L\sin\theta\cos^2\theta)}{2h}$$

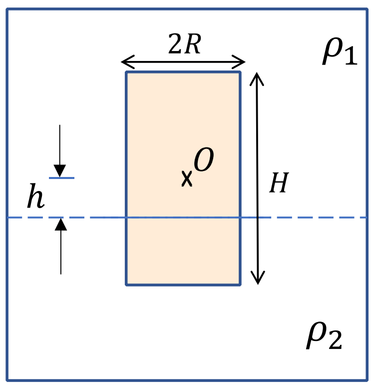
Then

$$\mu \geq \frac{f}{N_1} = \frac{L\sin^2\theta\cos\theta}{2h - L\sin\theta\cos^2\theta} = 0.35$$

[Another solution of this problem is obtained by noting that the three forces are concurrent. Then the maximum angle of limiting friction can be solved by trigonometry without having to consider the magnitude of the forces.]

6. A homogeneous cylinder of radius R and height H floats at the interface of two fluids with densities $\rho_1$ and $\rho_2$ respectively. If $\rho_2 = 4\rho_1$ and the density of the homogeneous cylinder is $\rho = 1.5\rho_1$. What is the distance $h$ from the center of mass of the cylinder to the interface between two fluids if the cylinder remains stationary?

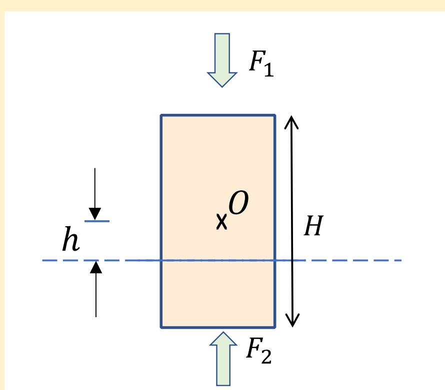
6. 半徑為 $R$ 、高度為 $H$ 的均質圓柱體浮在密度分別為 $\rho_1$ 和 $\rho_2$ 的分層液體界面處。設 $\rho_2 = 4\rho_1$，匀質圓柱體的質量 $\rho = 1.5\rho_1$。求當圓柱體保持靜止時，圓柱體重心與分層液體界面的距離 $h$ 為多大？




A. $\frac{H}{6}$ B. $\frac{H}{4}$ C. $\frac{H}{3}$ D. $\frac{H}{2}$ E. $\frac{2H}{3}$

Solution:

Let the volume of sphere immersed in fluid $\rho_1$ is $V_1$ and the downward (pressure) force due the fluid $\rho_1$ is $F_1$. The upward (pressure) force due the fluid $\rho_2$ is $F_2$.



$$V_1 = \pi R^2 \left(\frac{H}{2} + h\right)$$

$$V_2 = \pi R_2 \left(\frac{H}{2} - h\right)$$

Consider the equilibrium at the interface (if the cylinder is substituted by the fluid),

$$F_1 + \rho_1 V_1 g = \rho_1 g Y \pi R^2 \rightarrow F_1 = \rho_1 g \, \pi R^2 \left(Y - \frac{H}{2} - h\right)$$

where $Y$ is the depth of the fluid $\rho_1$. Similarly,

$$F_2 = \rho_1 g Y \pi R^2 + \rho_2 V_2 g \rightarrow F_2 = \rho_1 g \pi R^2 Y + \rho_2 g \pi R^2 \left(\frac{H}{2} - h\right)$$

If the cylinder is in equilibrium,

$$F_2 - F_1 = \rho g \pi R^2 H$$

We get

$$\rho_1 g \pi R^2 Y + \rho_2 g \pi R^2 \left(\frac{H}{2} - h\right) - \rho_1 g \, \pi R^2 \left(Y - \frac{H}{2} - h\right) = \rho g \pi R^2 H$$

$$\rightarrow \rho_2 \left(\frac{H}{2} - h\right) + \rho_1 \left(\frac{H}{2} + h\right) = \rho H$$

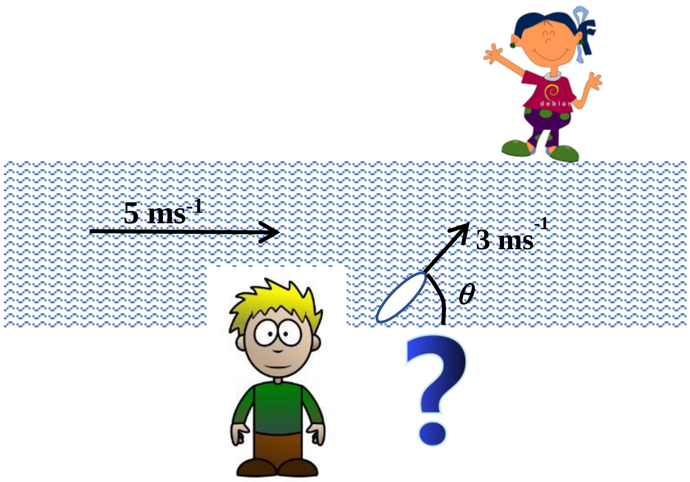
$$\rightarrow \frac{(\rho_2 + \rho_1) H}{2} - (\rho_2 - \rho_1) h = \rho H$$

$$\rightarrow \ 3\rho_1 h = \frac{5}{2} \rho_1 H - \frac{3}{2} \rho_1 H = \rho_1 H$$

$$\rightarrow h = \frac{H}{3}$$

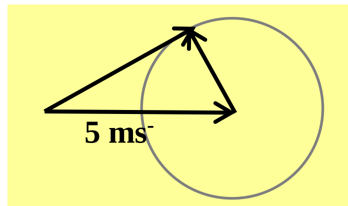
7. Peter is planning to sail a boat across a river. He can sail the boat at a speed of 3 $ms^{-1}$. The water in the river is flowing at the speed of 5 $ms^{-1}$. To arrive at the opposite bank of the river with the smallest distance downstream, what should be the angle $\theta$ between the sailing direction of his boat and the downstream direction of the river?

7. 彼得計劃乘船駛過一條河。他能以 3 $\mathrm{ms}^{-1}$ 的速度航行。河水以 5 $\mathrm{ms}^{-1}$ 的速度流動。要以最小下游距離到達河對岸，他的船的航行方向和下游方向之間的角度 $\theta$ 應該是多少？



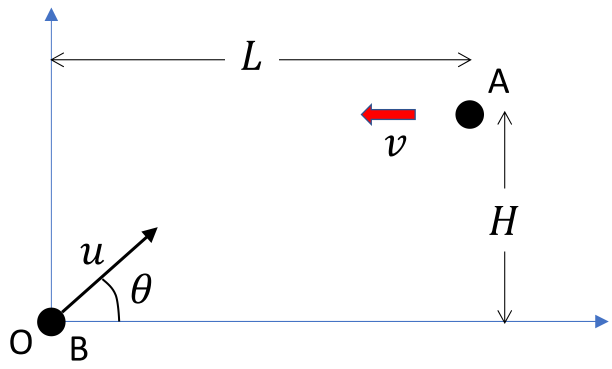

A. 90°　　B. 121°　　C. 127°
D. 143°　　E. 149°

To arrive at the opposite bank of the river with the smallest distance downstream, the resultant velocity of the boat has to make the largest angle with the river bank. This is obtained by having the resultant velocity tangential to the circle using the sailing velocity as the radius. Hence the angle is $90^{\mathrm{o}} + \arcsin 3/5 = 127^{\mathrm{o}}$. Answer: C




8. Ball A is projected horizontally with velocity $v = 1\ \mathrm{ms}^{-1}$ at the point $(L, H) = (20\mathrm{m}, 10\mathrm{m})$. To intercept the ball A, another ball B is projected at the origin with velocity $u = 5\ \mathrm{ms}^{-1}$. What is the launching angle $\theta$ of the ball B?

8. 球 A 在位置 $(L, H) = (20\mathrm{m}, 10\mathrm{m})$以速度 $v = 1\ \mathrm{ms}^{-1}$ 作水平投射。為截取球 A ，另一球 B 以速度 $u = 5\ \mathrm{ms}^{-1}$ 在原點投射。問球 B 的發射角 $\theta$ 是多少？




A. 20.7°  B. 26.6°  C. 31.7°  D. 33.9°  E. 35.3°

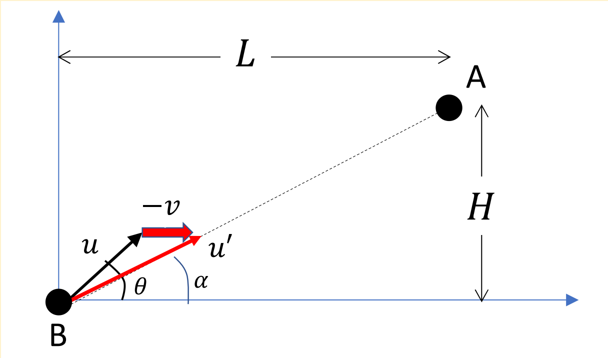
Solution:

The problem can be solved easily by transforming into the reference frame of the ball A. In this frame, the velocity of the B is $u'$ which is heading towards the A.

$$\tan\alpha = \frac{H}{L} \rightarrow \alpha = 26.565°$$

By sine law,

$$\frac{u}{\sin\alpha} = \frac{v}{\sin(\theta-\alpha)} \rightarrow \sin(\theta-\alpha) = \frac{v}{u}\sin\alpha \rightarrow \theta = 31.7°$$




9. Two friends Sandy and Lilian are playing a game of throwing stones. Lilian is at rest and Sandy is throwing a stone to Lilian once every $T$ seconds, while at the same time walking at a constant velocity $v$ away from Lilian (a negative $v$ means Sandy is walking towards Lilian). The speed of the stone in air is $u > |v|$. Lilian receives a stone every $T'$ seconds. Find $T'$.

9. 兩個朋友 Sandy 和 Lilian 正在玩扔石頭的遊戲。 Lilian 靜止不動，Sandy 每隔 $T$ 秒向 Lilian 投擲一塊石頭，同時以恆定速度 $v$ 遠離 Lilian（若 $v$ 值為負則表示 Sandy 正向 Lilian 走去）。 石頭在空中的速率是 $u > |v|$。 Lilian 每 $T'$ 秒接到一塊石頭。 求 $T'$。

A. $T' = T$

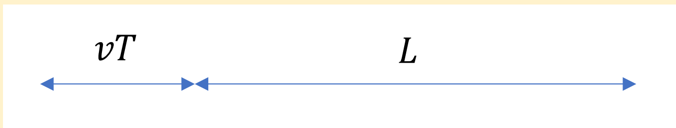
B. $T' = T\frac{u+v}{u}$

C. $T' = T\frac{u-v}{u}$

D. $T' = T\frac{u}{u+v}$

E. $T' = T\frac{u}{u-v}$

Solution:




The time between two stones $T' = T + \frac{v}{u}T = \left(\frac{u+v}{u}\right)T$

10. Following the previous question. This time Sandy is at rest and Lilian is walking towards Sandy with a constant velocity of $v$, where $|v| < u$ (a negative $v$ means Lilian is walking away from Sandy). Lilian receives a stone every $T'$ seconds. Find $T'$.

10. 承上題。 這次 Sandy 靜止不動，Lilian 以恆定速度 $v$ 向 Sandy 走去（若 $v$ 值為負則表示 Lilian 正在遠離 Sandy）。此處 $|v| < u$ 。 Lilian 每 $T'$ 秒接到一塊石頭。求 $T'$。

A. $T' = T$

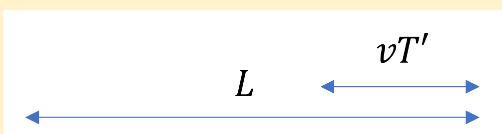
B. $T' = T\frac{u+v}{u}$

C. $T' = T\frac{u-v}{u}$

D. $T' = T\frac{u}{u+v}$

E. $T' = T\frac{u}{u-v}$

Solution:




The time between two stones is $T' = T - \frac{v}{u}T' \rightarrow T' = \frac{u}{u+v}T$

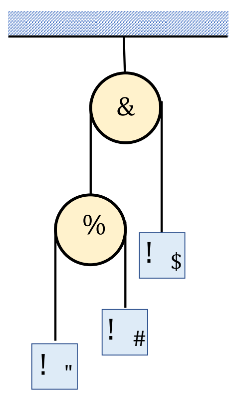
11. A massless rope is passed over a frictionless tree branch. A box of mass 12 kg is hung from one end, and a monkey of mass 10 kg is hung from the other end. What should the monkey do to lift up the box? (Choose all valid answers. Marks will only be given if all valid answers are the only chosen ones.)

11. 一根無質量繩索跨過無摩擦的樹枝。一端懸掛質量為12 kg 的盒子，另一端懸掛質量為 10 kg 的猴子。猴子應該做什麼來提升這個盒子？（請選擇所有正確答案，只有在所有正確答案都是唯一選定答案的情況下，才會給予分數。）

A. Climb up the rope with accelerating speed. 以加速的速度攀上繩索。
B. Climb up the rope with decelerating speed. 以減速的速度攀上繩索。
C. Climb down the rope with accelerating speed. 以加速的速度爬下繩子。
D. Climb down the rope with decelerating speed. 以減速的速度爬下繩子。
E. Climb with a constant speed. 以恆定速度攀登。

Answer: A and D.

12. As shown in the figure, masses $m_1 = m$ and $m_2 = 2m$ are connected by a string over a massless, frictionless pulley A. The axle of pulley A is connected by a second string over a second massless, frictionless pulley B to a mass $m_3 = 3m$. Pulley B is suspended from the ceiling by an attachment to the axle. The system is released from rest. What is the acceleration of the mass $m_1$?
12. 如圖所示，質量分別為 $m_1 = m$ 和 $m_2 = 2m$ 的物體由一條細線連接，細線跨過一個無質量、無摩擦的滑輪 A。滑輪 A 的軸心由另一條細線連接至質量為 $m_3 = 3m$ 的物體上，細線跨過第二個無質量、無摩擦的滑輪 B。滑輪 B 的軸心則懸掛在天花板上。系統從靜止中放開。問質量 $m_1$ 物體的加速度是多少？




A. $\frac{5}{17}g$ upward

B. $\frac{5}{17}g$ downward

C. $\frac{7}{17}g$ upward

D. $\frac{7}{17}g$ downward

E. $\frac{9}{17}g$ upward

Solution:

Let the tension on two strings be $T_A$ and $T_B$. And take all accelerations to be positive downward.

Since the pulley A is massless, we have

$$2T_A = T_B$$

$$m_1 g - T_A = m_1 a_1$$

$$m_2 g - T_A = m_2 a_2$$

$$m_3 g - T_B = m_3 a_3$$

Since the length of the strings are fixed, there is a constraint about the accelerations

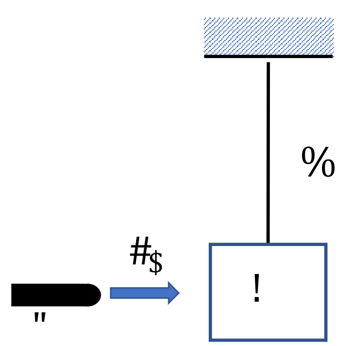
$$a_1 + a_2 + 2a_3 = 0$$

We have 5 equations and 5 unknowns and hence we can solve them and get,

$$a_1 = g\frac{4m_1m_2 - 3m_2m_3 + m_1m_3}{4m_1m_2 + m_2m_3 + m_1m_3} = -\frac{7}{17}g \text{ (upward)}$$

13. A block of mass $M = 1$ kg is attached to a massless rope of length $l = 1$ m which is initially at rest. A bullet with a mass of $m = 10$ g and a speed of $V_0 = 400\ \text{ms}^{-1}$ strikes the block horizontally. The bullet embeds itself there and the whole system swings. Find the maximum angle of the swinging.

13. 長度為 $l = 1$ m 的輕質懸線，下端繫上一質量為 $M = 1$ kg 的物塊，初始靜止。一顆質量為 $m = 10$ g 子彈以速度 $V_0 = 400\ \text{ms}^{-1}$ 水平射向物塊。子彈射入物塊後一起繞線上端懸掛點擺動。求最大擺動角。




A. 26° B. 39° C. 52° D. 64° E. 78°

Solution:

After embedding, the velocity of the system becomes

$$v = \frac{mV_0}{m+M}$$

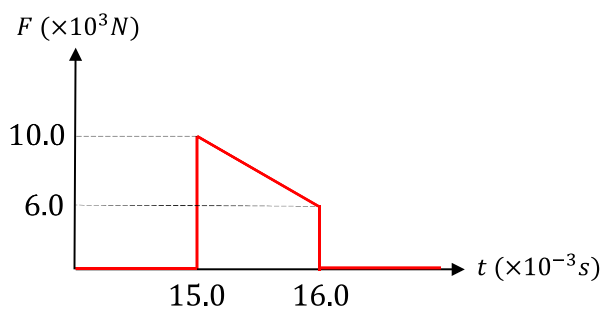
The conservation of energy gives,

$$\frac{1}{2}(m+M)v^2 = (m+M)gl(1-\cos\theta)$$

$$\rightarrow \cos\theta = 1 - \frac{1}{2gl}\frac{m^2V_0^2}{(m+M)^2} = 78°$$

14. A 2.00 kg stone is sliding to the right on a frictionless horizontal surface at 3.00 $\mathrm{ms^{-1}}$ when it suddenly collides with an object that exerts a large horizontal force to the left on it for a short period of time. The graph shows the magnitude of this force as a function of time. What is the magnitude and direction of the stone's velocity after collision?

14. 一個2.00 kg的石塊在無摩擦的水平面以3.00 $\mathrm{ms^{-1}}$的速度向右滑動，當它突然與一個物體發生碰撞時，該物體在短時間內施加一個向左的水平力於石塊上。下圖顯示了這個力作為時間的函數。問碰撞後石頭的速度和方向是什麼？




<u>A. 1 $\mathrm{ms^{-1}}$ to the left</u>　B. 1 $\mathrm{ms^{-1}}$ to the right　C. 7 $\mathrm{ms^{-1}}$ to the left

D. 7 $\mathrm{ms^{-1}}$ to the right　E. 5 $\mathrm{ms^{-1}}$ to the left

Solution:

The impulse on the stone is $J = \frac{1}{2}(16\times 10^3)(10^{-3}) = 8Ns = -\Delta p = mv_f - mv_i$

$$v_f = v_i - \frac{\Delta p}{m} = 3 - \frac{8}{2} = -1\ \mathrm{ms^{-1}}$$

15. A 1 kg ball is released from rest at a height of 20 m above ground at $t = 0$. After 1 s, a 2 kg ball is thrown downward with an initial speed of $20\ \mathrm{ms^{-1}}$. Find the time $t = T$ when the center of mass of the two balls first comes to rest. The collision of the balls and the ground is perfectly inelastic, whereas the collision between the two balls (if they do collide) is elastic.

15. 在 $t = 0$ 時，一個 1 kg 的靜止的球從離地面 20 m 的高處開始下墜。 1 秒後，又以 $20\ \mathrm{ms^{-1}}$ 的初始速度向下拋出另一 2 kg 的球。找出兩個球的質心到達靜止狀態所需的時間 $t = T$。 球與地面的碰撞是完全非彈性的，而兩個球之間的碰撞（如果它們碰撞的話）是彈性碰撞。

A. 2 s　　B. 1.83 s　　C. 1.79 s
<u>D. 1.93 s</u>　　E. 0.93 s

Solution:
At time $t' = t - 1 \geq 0$,

$$y_1(t') = 20 - \frac{1}{2} g (t'+1)^2, \qquad v_1(t') = -g(t'+1)$$

$$y_2(t') = 20 - 20t' - \frac{1}{2} g t'^2, \qquad v_2(t') = -20 - gt'$$

Two balls will collide (i.e. $y_1 = y_2$) at time $t'$, where,

$$-gt' - \frac{1}{2} g = -20t' \rightarrow t' = \frac{g}{2(20-g)} = 0.441s$$

At that time, $y_1(t') = 9.825\ m$, $v_1 = -14.1218\ m/s$ and $v_2 = -24.3218\ m/s$.

Next, we need to find the velocity of ball 2 after collision.

$$u_2 = \frac{m_2 - m_1}{m_2 + m_1} v_2 + \frac{2m_1}{m_2 + m_1} v_1 = -17.5218\ m/s$$

The additional time for it to hit on the ground is

$$0 - 9.825 = -17.5218\ t'' - \frac{1}{2} g t''^2 \rightarrow t'' = 0.493\ s$$

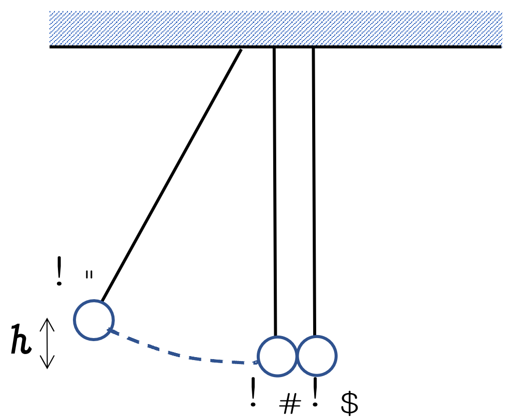
The total time is $t = 1 + t' + t'' = 1.93$ s.

16. Three small balls of the same size but different masses are hung side-by-side in parallel on the strings of same length. They touch each other. The first ball of mass $m_1$ is pulled to the height $h$ and released from rest. The first ball collides elastically with the second ball of mass $m_2$ and the second ball collides elastically with the third ball of mass $m_3$ afterwards. After these two collisions,

three balls will have the same momentum. What is the value of $m_3$?

16. 三個大小相同，質量不同的小球，並排平行地懸掛在同樣長度的線上，彼此相互接觸。今把質量為 $m_1$ 的第一個小球拉開，上升到高度為 $h$ 處，再自由放開。第一個小球與質量為 $m_2$ 的第二個小球作彈性碰撞，接著第二個小球與質量為 $m_3$ 的第三個小球作彈性碰撞。經這兩次碰撞後，三球具有相同動量，$m_3$ 的值是多小？

A. $\frac{1}{6}m_1$ B. $\frac{1}{3}m_1$ C. $\frac{1}{2}m_1$ D. $\frac{2}{3}m_1$ E. $m_1$




Solution:

Before the collision, the momentum of $m_1$ is $p_0 = m_1 u_1 = m_1\sqrt{2gh}$

After the first collision, the momentum of mass 1 and 2 are $p_1$ and $p_2$ respectively.

$$p_1 + p_2 = p_0$$

$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{p_0^2}{2m_1}$$

$$p_1 = \frac{m_1 - m_2}{m_1 + m_2}p_0 \quad and \quad p_2 = \frac{2m_2}{m_1 + m_2}p_0$$

After the second collision, the momentum of mass 2 and 3 are $p_2{}'$ and $p_3'$, we get

$$p_2' = \frac{m_2 - m_3}{m_2 + m_3}p_2 = \left(\frac{m_2 - m_3}{m_2 + m_3}\right)\left(\frac{2m_2}{m_1 + m_2}\right)p_0$$

$$p_3' = \frac{2m_3}{m_2 + m_3}p_2 = \left(\frac{2m_3}{m_2 + m_3}\right)\left(\frac{2m_2}{m_1 + m_2}\right)p_0$$

To have the same momentum, $p_1 = p_2' = p_3'$

$$m_3 = \frac{1}{3}m_2$$

and

$$\left(\frac{m_2 - m_3}{m_2 + m_3}\right)\left(\frac{2m_2}{m_1 + m_2}\right)p_0 = \frac{m_1 - m_2}{m_1 + m_2}p_0 \to \frac{2}{4} \times 2m_2 = m_1 - m_2 \to m_2 = \frac{m_1}{2}$$

$$\to m_3 = \frac{1}{6}m_1$$

17. An astronaut lands on a spherical planet in a distant galaxy. He throws a 2.50 kg stone upward from the ground at 12.0 $\mathrm{ms^{-1}}$ and it returns to the ground at 6.00 s. The circumference of the planet is $2.00 \times 10^5$ km and there is no atmosphere on the planet. If a spaceship goes into a circular orbit 30 000 km above the surface of the planet, how many hours will it take the ship to complete one orbit?

17. 宇航員登上了遙遠星系內的一顆球形行星。他從地面以速率 12.0 $\mathrm{ms^{-1}}$向上拋出一塊 2.50 kg 的石頭，石頭在 6.00 s 後回到地面。行星的周長是 $2.00 \times 10^5$ km ，而且行星上沒有大氣。如果一艘太空船進入距離行星表面 30 000 km 的圓形軌道，那麼這艘船需要多少小時完成一個軌道？

A. 11.3 hours　　B. 12.7 hours　　C. 13.3 hours (underlined)
D. 14.5 hours　　E. 15.6 hours

Solution:
For constant acceleration,

$$a = \frac{v - v_0}{t} = \frac{24\ ms^{-1}}{6\ \mathrm{s}} = 4.00\ \mathrm{ms^{-2}}$$

The radius of the planet is $R = \frac{C}{2\pi} = 3.18 \times 10^7$ m.
The mass of the planet is

$$M = \frac{aR^2}{G} = 6.06 \times 10^{25}\ \mathrm{kg}$$

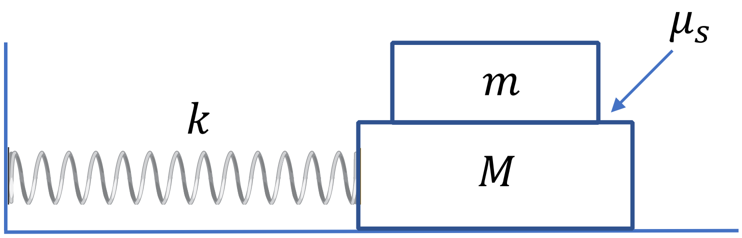
In the circular orbit

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \to \text{period } T = \frac{2\pi r}{v} = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{GM}} = 4.80 \times 10^4\ s = 13.3\ h$$

18. - 19. A block of mass $M = 1$ kg rests on a frictionless surface and is connected to a horizontal

spring of force constant $k = 50$ N/m. The other end of the spring is attached to a wall. A second block with mass $m = 0.2$ kg rests on top of the first block. The coefficient of static friction between the blocks is $\mu_s = 0.2$.

18.－19. 一塊質量為 $M = 1$ kg的物塊放在無摩擦的平面上，並連接到一個力常數 $k =$ 50 N/m 彈簧的另一端連接到牆上。第二個質量為 $m = 0.2$ kg的物塊放於第一塊的頂部。物塊之間的靜摩擦係數是 $\mu_s = 0.2$。




18. Find the maximum amplitude of oscillation such that the top block will not slip on the bottom block.

18. 找出擺動的最大振幅，以使上方物塊不會在下方物塊上滑動。

A. 0.8 cm　　B. 3.9 cm　　C. 4.0 cm
<u>D. 4.7 cm</u>　　E. 5.4 cm

19. If the spring is compressed to $A = 10$ cm and the system is released at rest initially. When will the top block slip on the bottom block?

19. 如果彈簧被壓縮至 $A = 10$ cm 並且系統最初處於靜止狀態，上方物塊在什麼時候開始在下方物塊上滑動？

A. 0.03 s　　<u>B. 0.17 s</u>　　C. 0.34 s
D. 4.8 s　　E. 9.6 s

Solution:

Consider the entire system, we have

$$(m + M)\ddot{x} = -kx$$

and

$$x(t) = A\cos(\omega t + \phi)$$

where $\omega = \sqrt{\frac{k}{m+M}} = 6.45\ s^{-1}$.

Next, we consider solely the top block,

$$\mu_S mg \geq f_s = m\ddot{x} = \frac{mkx}{m+M}$$

Hence, the maximum amplitude is

$$A = \frac{\mu_s(m+M)g}{k} = 0.047 \text{ m} = 4.7 \text{ cm}$$

Now, if $A = 10 \text{ cm} = 0.1 \text{ m}$,

$$x(t) = 0.1\cos(\omega t)$$

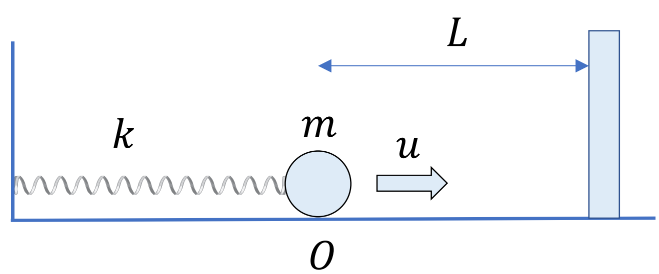
The top block will slide when

$$\frac{mkx(t)}{m+M} = \mu_s mg \rightarrow x(t) = \frac{\mu_s(m+M)g}{k} = 0.047 \text{ m}$$

$$\cos \omega t = 0.47 \rightarrow 0.17 \text{ s}$$

20. A small ball of mass $m$ rests on a frictionless surface and is connected to a horizontal spring of force constant $k$. The other end of the spring is attached to a wall. At time $t = 0$, the spring is unstretched and the ball is moving to right with initial speed $u$ at the origin. There is another wall at distance $L$ from the origin where $L^2 < \frac{4mu^2}{k}$. The collisions between the ball and the wall is elastic. Find the time when the ball will return to the origin for the first time.

20. 一個質量為 $m$ 的小球放在無摩擦的平面，並連接到一個彈力係數為 $k$ 的水平彈簧上。彈簧的另一端連接到牆上。在 $t = 0$ 時，彈簧是未拉伸的，而球在原點處以初始速度 $u$ 向右移動。在距離原點 $L$ 的地方有另一面牆，且 $L^2 < \frac{4mu^2}{k}$。球和牆之間的碰撞是彈性碰撞。求小球首次回到原點的時間。



A. $t = \pi\sqrt{\frac{m}{k}}$

B. $t = \frac{1}{2}\sqrt{\frac{m}{k}}\frac{1}{\cos^{-1}\left(\frac{L}{u}\sqrt{\frac{k}{m}}\right)}$

C. $t = \frac{1}{2}\sqrt{\frac{m}{k}}\frac{1}{\sin^{-1}\left(\frac{L}{u}\sqrt{\frac{k}{m}}\right)}$

D. $t = 2\sqrt{\frac{m}{k}}\cos^{-1}\left(\frac{L}{u}\sqrt{\frac{k}{m}}\right)$

E. $t = 2\sqrt{\frac{m}{k}}\sin^{-1}\left(\frac{L}{u}\sqrt{\frac{k}{m}}\right)$

Solution:
Without the wall, the trajectory of the small ball is

$$x(t) = A\sin(\omega t)$$

where $\omega = \sqrt{\frac{k}{m}}$. The initial velocity of the ball is $u$, hence $A = \frac{u}{\omega} = u\sqrt{\frac{m}{k}}$.
The ball will hit the wall at time $t$, where

$$L = u\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right) \to t = \sqrt{\frac{m}{k}}\sin^{-1}\left(\frac{L}{u}\sqrt{\frac{k}{m}}\right)$$

The total time for the ball to return is

$$2t = 2\sqrt{\frac{m}{k}}\sin^{-1}\left(\frac{L}{u}\sqrt{\frac{k}{m}}\right)$$

**Open Problems (12 Marks Each) 開放題(每題 12 分)**

1. The trebuchet is a medieval weapon of war – a catapult that is powered by a falling massive counterweight and a swinging arm to throw a projectile. In this problem, we analyze a simplified version of the trebuchet – the see-saw trebuchet (Fig. 1) It has appeared in movies such as The Lord of the Rings: The Return of the King.

1. 投石機是一種中世紀的戰爭武器 – 一種由落下的大型平衡重和擺動臂來投擲拋射物的彈射器。在這個問題中，我們分析一個簡化版本的投石機 – 蹺蹺板投石機（圖 1）。它曾出現在電影中，如魔戒三部曲:王者再臨 。

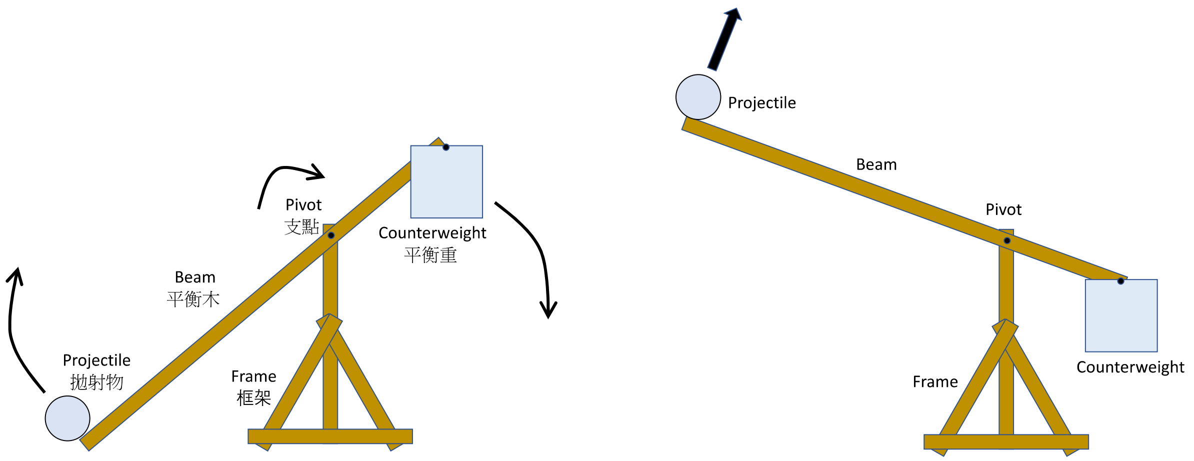
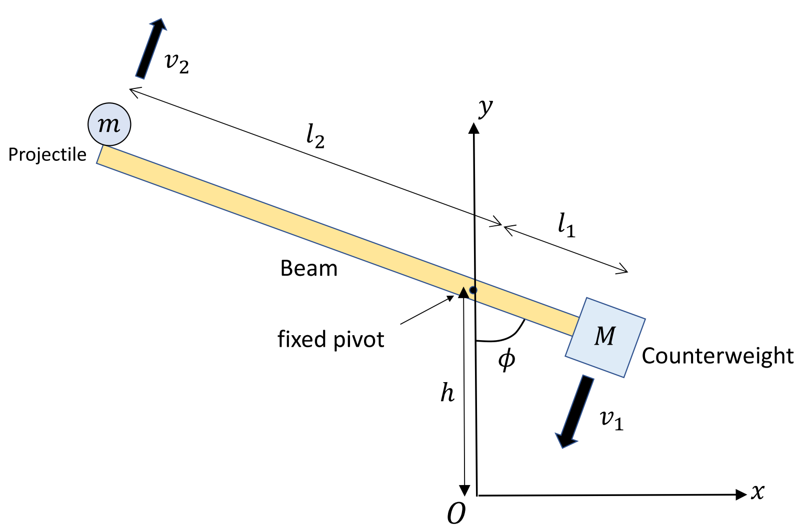



Figure 1: Systematic picture of a see-saw trebuchet
圖 1：一個蹺蹺板投石機的圖片




In this problem, we fix the pivot at the height $h = 2$ m above the origin $O$. And here are the parameters you will need:

在這個問題中，我們將支點固定在原點上方 $h = 2$ m 的高度。以下是您需要的參數：

| | |
|---|---|
| $m = 1$ kg | mass of projectile 拋射物的質量 |
| $M = 500$ kg | mass of counterweight 平衡重的質量 |
| $l_1 = 0.5$ m | distance from pivot to counterweight 從支點到平衡重的距離 |
| $l_2 = 4$ m | distance from pivot to projectile 從支點到拋射物的距離 |
| $v_1$ | speed of counterweight 平衡重的速率 |
| $v_2$ | speed of projectile 拋射物的速率 |
| $\phi$ | beam angle 平衡木的角度 |

To simplify the analysis, assume that the beam is massless. The counterweight and projectile are treated as point masses, and both are released from rest at the initial beam angle $\phi_0 = 120°$.
為了簡化分析，假定平衡木是無質量的。平衡重和拋射物可視為點質量，並且都是在平衡木的初始角度 $\phi_0 = 120°$ 時從靜止中放開。

(a) Find the relationship between the speed of the counterweight ($v_1$) and the projectile ($v_2$) at angle $\phi$.
（a）找出平衡重($v_1$)和拋射物($v_2$)在角度$\phi$時的速度之間的關係。

Solution:
If the angular velocity of the beam is $\omega$, we have

$$v_1 = l_1\omega \quad \text{and} \quad v_2 = l_2\omega$$

$$\rightarrow \frac{v_1}{v_2} = \frac{l_1}{l_2} = \frac{1}{8}$$

(b) Calculate the total mechanical energy of the system at beam angle $\phi$ in terms of $M, m, l_1, l_2$ and $v_2$. Assuming the gravitational potential energy of masses are zero at the initial beam angle $\phi_0$.
(b)計算平衡木在角度 $\phi$ 時的總機械能，以 $M, m, l_1, l_2$ 和 $v_2$ 來表達。假設物體的引力勢能在平衡木的初始角度為零。

Solution:
Initial height of the projectile = $l_2 \cos\phi_0 = -l_2/2$
Initial height of the counterweight = $-l_1 \cos\phi_0 = l_1/2$

final height of the projectile = $l_2 \cos\phi$
final height of the counterweight = $-l_1 \cos\phi$

The total mechanical energy at beam angle $\phi$ is

$$E=\frac{1}{2}Mv_1^2+\frac{1}{2}mv_2^2+Mg\left(-l_1\cos\phi-\frac{l_1}{2}\right)+mg(l_2\cos\phi+\frac{l_2}{2})$$
$$E=\frac{1}{2}\left(M\left(\frac{l_1}{l_2}\right)^2+m\right)v_2^2-Mgl_1\left(\cos\phi+\frac{1}{2}\right)+mgl_2(\cos\phi+\frac{1}{2})$$

(c) If the projectile is launched at the beam angle $\phi=45°$, what is the numerical value of the initial speed ($v_2$) of the projectile?
(c)如果拋射物在平衡木的角角為 $\phi=45°$ 時發射，問拋射物的初始速度($v_2$)的數值是多少？

Solution:
By the conservation of energy, we have
$$\frac{1}{2}\left(M\left(\frac{l_1}{l_2}\right)^2+m\right)v_2^2=(Ml_1-ml_2)\left(\cos\phi+\frac{1}{2}\right)g$$
$$v_2=\sqrt{\frac{2(Ml_1-ml_2)\left(\cos\phi+\frac{1}{2}\right)g}{\left(M\left(\frac{l_1}{l_2}\right)^2+m\right)}}=25.7\ \mathrm{ms^{-1}}$$

(d) What is the numerical value of the horizontal range, $d$, of the projectile relative to the origin?
(d)求拋射物相對於原點的水平範圍 $d$ 的數值是多少？

Solution:
The launch angle of the projectile is $\phi$ and the initial position of the projectile is $(x,y)=(-l_2\sin\phi, l_2\cos\phi+h)$.

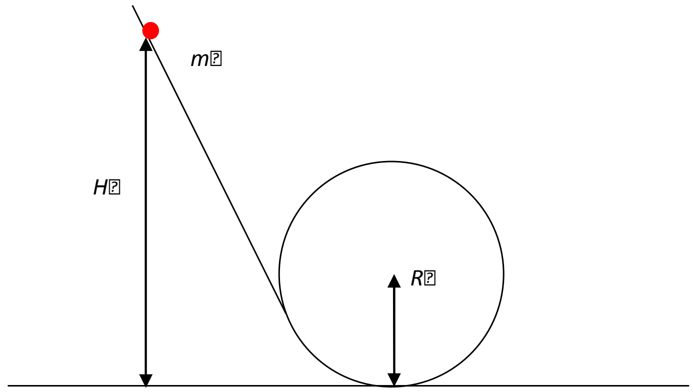
To hit the ground, the y-component is zero.

$$0=l_2\cos\phi+h+v_2\sin\phi\, t-\frac{1}{2}gt^2$$
$$\rightarrow t=3.96\ \mathrm{s}$$

And the horizontal range is
$$x=-l_2\sin\phi+v_2\cos\phi\, t=69\ \mathrm{m}$$

2. An object with mass *m* is initially at rest at a distance *H* above ground on a smooth track, as shown in the figure below.

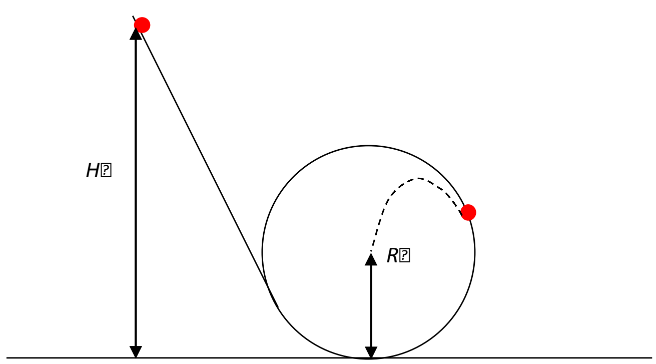
2. 如下圖所示，質量為 $m$ 的物體最初靜止在平滑軌道上距離地面 $H$ 處。




The lower part of the track is circular and has radius $R$. It is assumed that all frictional forces can be ignored.
軌道的下部為圓形，半徑為 $R$。假設所有摩擦力都可以忽略。

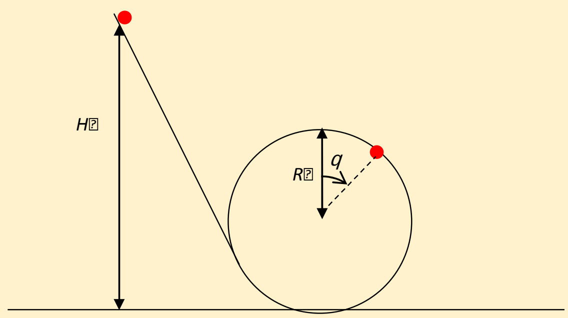
(a) Find the critical initial height $H_c$ below which the object cannot complete the whole journey along the circular track.
(a) 求初始高度的臨界值 $H_c$。若初始高度低於該臨界高度，則物體不能沿著圓形軌道完成整個行程。

(b) When $H < H_c$, the object will leave the track at some point. Find $H$ so that the object will hit the center of the circle after leaving the track.
(b) 當 $H < H_c$ 時，物體將在某個點離開軌道。求 $H$，使得物體在離開軌道後將撞到圓心。



Solution:

(a) Measure the angular position of the object by the angle $\theta$ defined in the figure below.




Considering the centripetal force, we have

$$N + mg\cos\theta = \frac{mv^2}{R}$$

where $N$ is the normal reaction of the track.

The object will leave the track when $N = 0$. One can see that this is possible only when $-90^\circ \le \theta \le 90^\circ$.

By conservation of energy, the speed $v$ is determined by

$$\frac{1}{2}mv^2 + mgR\left(1+\cos\theta\right) = mgH\ .$$

Solving the two equations for $N$, we have

$$N = -mg\cos\theta + \frac{2mgH - 2mgR\left(1+\cos\theta\right)}{R} = \frac{2mgH}{R} - mg\left(3\cos\theta + 2\right)$$

Hence the angular position of the point the object loses contact is determined by

$$\frac{2mgH}{R} - mg\left(3\cos\theta + 2\right) = 0 \Rightarrow \cos\theta = \frac{2}{3}\left(\frac{H}{R} - 1\right),$$

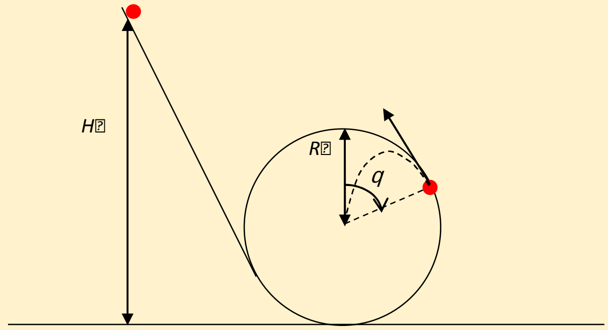
where $-90^\circ \le \theta \le 90^\circ \rightarrow \cos\theta \ge 0$.

One can see that there will be no solution when

$$\frac{2}{3}\left(\frac{H}{R} - 1\right) > 1 \Rightarrow H > \frac{5}{2}R.$$

Hence $H_c = 5R/2$. If $H < H_c$, the object will lose contact at some point.

(b) After the object leaves the track, it will perform projectile motion. By geometry, the launching angle is $\theta$.




Hence the trajectory is

$$y = R(1+\cos\theta) + x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta}.$$

The coordinate of the center is $(R\sin\theta, R)$.

If the center is on the trajectory

$$R = R(1+\cos\theta) + R\sin\theta\tan\theta - \frac{gR^2\sin^2\theta}{2v^2\cos^2\theta}$$

$$R\cos\theta + R\frac{\sin^2\theta}{\cos\theta} - \frac{gR^2\sin^2\theta}{2v^2\cos^2\theta} = 0$$

$$\cos^2\theta + \sin^2\theta - \frac{gR\sin^2\theta}{2v^2\cos\theta} = 0$$

$$gR\sin^2\theta = 2v^2\cos\theta$$

By energy conservation, $\frac{1}{2}mv^2 + mgR(1+\cos\theta) = mgH$ , we have

$$gR\sin^2\theta = 4\left[gH - gR(1+\cos\theta)\right]\cos\theta$$

$$R(1-\cos^2\theta) = 4\left[H - R(1+\cos\theta)\right]\cos\theta$$

$$R\left\{1-\left[\frac{2}{3}\left(\frac{H}{R}-1\right)\right]^2\right\} = 4\left[H - R\left(1+\frac{2}{3}\left(\frac{H}{R}-1\right)\right)\right]\frac{2}{3}\left(\frac{H}{R}-1\right)$$

$$12(H-R)^2 = 9R^2$$

$$H - R = \pm\frac{\sqrt{3}}{2}R$$

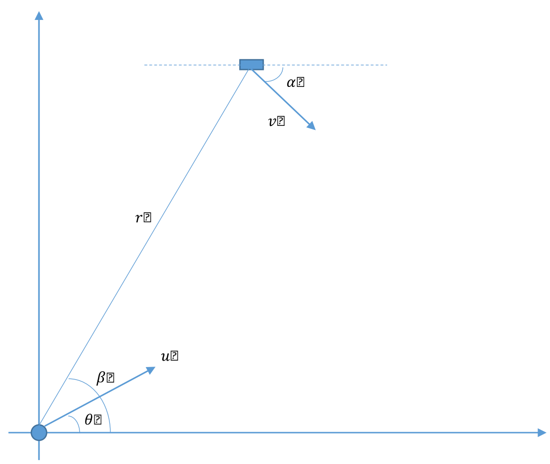
$$H = \left(1\pm\frac{\sqrt{3}}{2}\right)R$$

The solution $H = \left(1-\frac{\sqrt{3}}{2}\right)R < R$ is rejected because in this case $90° < \theta < 270°$ and the object will never leave the track.

Hence the solution is $H = \left(1 + \frac{\sqrt{3}}{2}\right)R < \frac{5}{2}R$.

3. An artificial satellite falls back to the Earth due to malfunction. During its last stage just before hitting the ground, its height is much smaller than the Earth's radius. To avoid its hitting the ground, a projectile is launched to destroy it, as shown in the figure below. The launching speed $u$ is fixed but the launching angle $\theta$ is arbitrary. Ignore air resistance.

3. 一人造衛星因故障墜向地球。 在撞擊地面之前的最後階段，其高度遠遠小於地球的半徑。 為了避免它撞到地面，會發射一枚拋射物來摧毀它，如下圖所示。 。




(a) If $r = 3$ km, $\beta = 60^\circ$, $v = 500\ \mathrm{ms^{-1}}$, $\alpha = 30^\circ$, find the time $T$ taken for the satellite to hit the ground if it were not hit by the projectile.

(a)設 $r = 3$ km，$\beta = 60^\circ$，$v = 500\ \mathrm{ms^{-1}}$，$\alpha = 30^\circ$，如果拋射物不能擊中衛星，求衛星撞擊地面的時間。

(b) Find the minimum value of $u$ and the corresponding launching angle $\theta$ so that the projectile can hit the satellite before it hits the ground.

(b) 計算 $u$ 的最小值和其對應的發射角 $\theta$，使得拋射物能在衛星撞擊地面之前擊中它。

Solution:

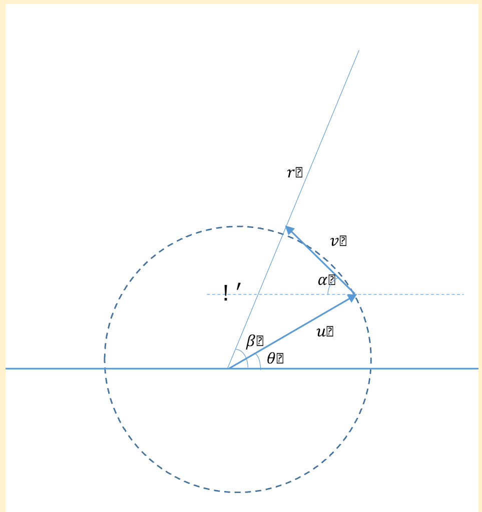
(a) Take downward to be positive

$$r \sin\beta = v \sin\alpha\ T + \frac{1}{2} g T^2$$

Taking the +ve solution of $T$, we have

$$\rightarrow T = \frac{-v\sin\alpha + \sqrt{v^2\sin^2\alpha + 2gr\sin\beta}}{g} = 8.855 \text{ s}$$

(b) Since there is no relative acceleration, the simplest way is to use relative velocity.




From the figure, we can see that

$$u\sin(\beta - \theta) = v\sin(\alpha + \beta)$$

$$\Rightarrow \theta = \beta - \sin^{-1}\left(\frac{v}{u}\sin(\alpha+\beta)\right) = 60^\circ - \sin^{-1}\frac{v}{u}$$

The magnitude of the relative velocity,

$$u' = \sqrt{u^2 + v^2 - 2uv\cos(\alpha + \theta)} = \sqrt{u^2 - v^2}$$

Alternatively,

$$uT\cos\theta = r\cos\beta + vT\cos\alpha \Rightarrow T = \frac{r\cos\beta}{u\cos\theta - v\cos\alpha}$$

$$uT\sin\theta - \frac{1}{2}gT^2 = r\sin\beta - vT\sin\alpha - \frac{1}{2}gT^2 \Rightarrow T = \frac{r\sin\beta}{u\sin\theta + v\sin\alpha}$$

Therefore,

$$\frac{r\cos\beta}{u\cos\theta - v\cos\alpha} = \frac{r\sin\beta}{u\sin\theta + v\sin\alpha}$$

$$u(\sin\beta\cos\theta - \cos\beta\sin\theta) = v(\sin\beta\cos\alpha + \cos\beta\sin\alpha)$$

$$\Rightarrow u\sin(\beta - \theta) = v\sin(\alpha + \beta)$$

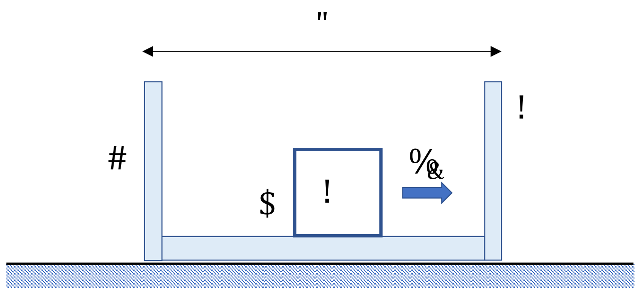
To get the minimum value of $u$, the projectile should hit the satellite at time $T$,

$$\frac{r}{u'} = T \rightarrow \sqrt{u^2 - v^2} = 338.79$$

$$\rightarrow u = 604\ ms^{-1}$$

4. There is a box A of length $L = 1.0$ m on the smooth horizontal surface. The center of mass of the box is located at its middle position. Inside the box, there is a block B with negligible size. Box A and block B have the same mass. Initially, box A is at rest and block B is located at the middle of A and moves to the right with an initial speed $v_0 = 5\ \mathrm{ms^{-1}}$. The coefficient of friction between A and B is $\mu = 0.05$ and the collisions between the left and right walls of A and B are elastic.

4. 在光滑水平地面上有一長為 $L = 1.0$ m 的箱子 A，箱子的質心位於正中。箱內有一物塊B(長度不計)， A 和 B 質量相等。初始時箱A靜止，物塊B位於A的正中以初速 $v_0 = 5\ \mathrm{ms^{-1}}$ 向右運動。A 和 B 的摩擦係數是 $\mu = 0.05$。假設 B 和 A 的左右兩壁的碰撞都是彈性碰撞。




(a) How many collisions occur between the walls of box A and block B?

(a) 物塊 B 與 A 的箱壁發生多少次碰撞？

(b) What is the horizontal displacement of box A from the initial moment to the moment when block B has just reached the relatively stationary position inside the box?

(b) 從初始瞬間起，到物塊 B 在箱內剛達到相對靜止的瞬間，箱子 A 在水平面上的位移是多少？

Solution:

(a) When the block is at rest relative to the box, they have the same (final) velocity $v$ relative to the lab frame. Since there are no external forces, the total momentum is conserved and we have,

$$2mv = mv_0 \to v = \frac{v_0}{2}$$

Since the box and the block have the same mass and the collision is elastic, the relative velocity of the block relative to the box change sign after every collision and the block decelerates between collisions due to the friction.

Hence, the total distance travelled by the block before it stops (relative to the box) is,

$$\mu mgs = \frac{1}{2}mv_0^2 - 2 \times \frac{1}{2}mv^2 \rightarrow s = \frac{v_0^2 - 2v^2}{2\mu g} = \frac{v_0^2}{4\mu g} = 12.76 \text{ m}$$

The block collides with the box at distances 0.5 m, 1.5 m, ..., 12.5 m. Hence the number of collisions is 13.

(b) Since there are no external forces acting on the system, the center of mass (CM) always moves with the constant velocity $v$. Hence the distance travelled by the CM is

$$x_{CM} = vt = \frac{v_0 t}{2}$$

where $t$ is the time for the block to decelerate from $v_0$ to $v$.

Consider the frictional force acting on the block, we have

$$-\mu mgt = mv - mv_0 = -\frac{1}{2}mv_0 \rightarrow t = \frac{v_0}{2\mu g} = 5.1 \text{ s}$$

And

$$x_{CM} = \frac{v_0 t}{2} = 12.76 \text{ m}$$

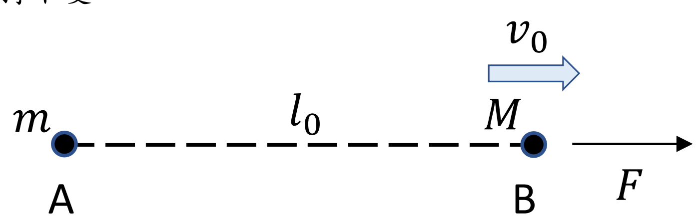
Initially, B is at the midpoint of A. At distances 0.5 m, 2.5 m, ..., 12.5 m, the block collides with the right wall of the box. Hence when the block stops, it lies at 0.26 m from the right wall. So finally it will stop at 0.24m to the right of the midpoint. Hence the CM is shifted to 0.12m to the right of the midpoint when the block B is at rest relative to the box.

The distance of the box travelled is

$$x_A = x_{CM} - 0.12 = 12.64 \text{ m}$$

5. Assume that there are two point particles A and B in space with masses $m$ and $M$ respectively. Initially, the separation between A and B is $l_0$. A is at rest and B moves with initial velocity $v_0$ along the line AB. An external force $F$ is applied to B in the direction of its motion such that the velocity of B is maintained at a constant.

5. 假設宇宙空間中有兩質點 A 和 B，它們的質量分別為 $m$ 和 $M$。開始時，A 和 B 相距為 $l_0$，A 靜止，B 以初速 $v_0$ 沿 AB 連線方向運動。有一外力 $F$ 沿著 B 的運動方向作用於其上，令它速度維持不變。




(a) If $v_0 < \sqrt{\frac{2GM}{l_0}}$, what is the magnitude of the external force $F$ when the separation between A and B is maximum. (Hint: you should consider the inertia reference frame where B is at rest.)
(a) 假設 $v_0 < \sqrt{\frac{2GM}{l_0}}$，試求 A, B 間距離最大時，外力 $F$ 的值。(提示：你可以考慮一個 B 是靜止的慣性參考系。)

(b) Calculate the total work done by the external force $F$ from the beginning to the time when two particles are maximally separated.
(b) 計算從初始瞬間到 A, B 間距離最大時，外力 $F$ 做的功。

(c) If $v_0 > \sqrt{\frac{2GM}{l_0}}$, calculate the total work done by the external force $F$ from the beginning to the time when two particles are maximally separated.
(c) 假設 $v_0 > \sqrt{\frac{2GM}{l_0}}$，計算從開始起到 A, B 間距離最大時，外力 $F$ 做的功。

Solution:

(a) In the inertial frame where mass A is at rest initially, both A and B are moving and it is difficult to determine to maximum separation between two masses. Instead, we consider another inertial frame where B is at rest. The initial velocity of A is $v_0$ and the only force acting on A is the gravity. By conservation of energy, we have

$$\frac{1}{2}mv_0^2 - \frac{GMm}{l_0} = -\frac{GMm}{l_m}$$

Hence the maximum separation between two masses is

$$l_m = \frac{GM}{\frac{GM}{l_0} - \frac{1}{2}v_0^2} = \frac{2l_0 GM}{2GM - l_0 v_0^2}$$

The external force $F$ acting on B at this point is

$$F = \frac{GMm}{l_m^2} = \frac{(2GM - l_0 v_0^2)^2 m}{4l_0^2 GM}$$

(b) We transformed back to the original inertial frame. When the separation between A and B is $l_m$, they both moves with the speed $v_0$. Hence, the work done is

$$W = \left(\frac{1}{2}mv_0^2 + \frac{1}{2}Mv_0^2 - \frac{GMm}{l_m}\right) - \left(\frac{1}{2}Mv_0^2 - \frac{GMm}{l_0}\right)$$

$$W = mv_0^2$$

(c) If $v_0 > \sqrt{\frac{2GM}{l_0}}$, the maximum separation between two masses is infinite. In the co-moving inertia frame (where B is at rest), we have

$$\frac{1}{2}mv_0^2 - \frac{GMm}{l_0} = \frac{1}{2}mv'^2_A \rightarrow v'_A = \sqrt{v_0^2 - \frac{2GM}{l_0}}$$

Return back to the original inertial frame,

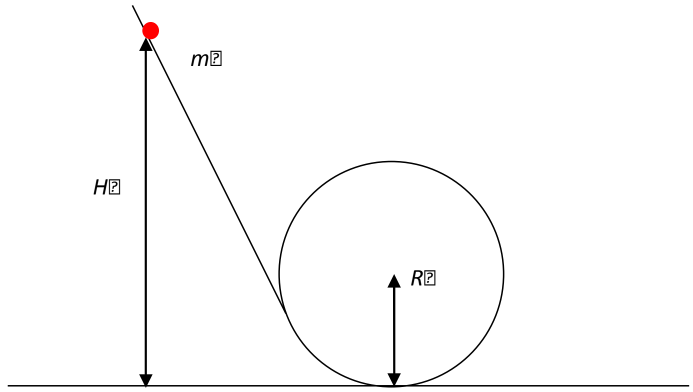
$$v_A = v_0 - v'_A = v_0 - \sqrt{v_0^2 - \frac{2GM}{l_0}}$$

when they are infinite apart. And the total work done by the external force is

$$W = \left(\frac{1}{2}mv_A^2 + \frac{1}{2}Mv_0^2\right) - \left(\frac{1}{2}Mv_0^2 - \frac{GMm}{l_0}\right) = mv_0\left(v_0 - \sqrt{v_0^2 - \frac{2GM}{l_0}}\right)$$

5. An object with mass $m$ is initially at rest at a distance $H$ above ground on a smooth track, as shown in the figure below.

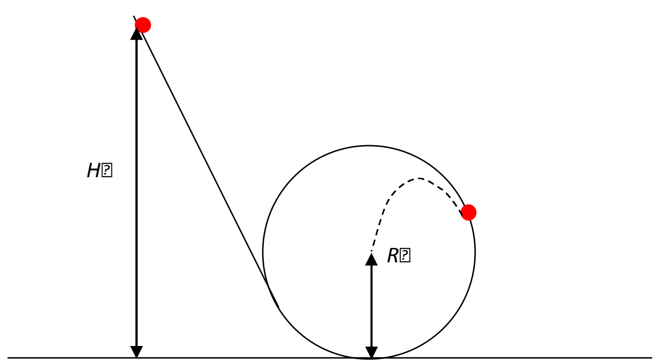
5. 如下圖所示，質量為 $m$ 的物體最初靜止在平滑軌道上距離地面 $H$ 處。




The lower part of the track is circular and has radius $R$. It is assumed that all frictional forces can be ignored.
軌道的下部為圓形，半徑為 $R$。假設所有摩擦力都可以忽略。

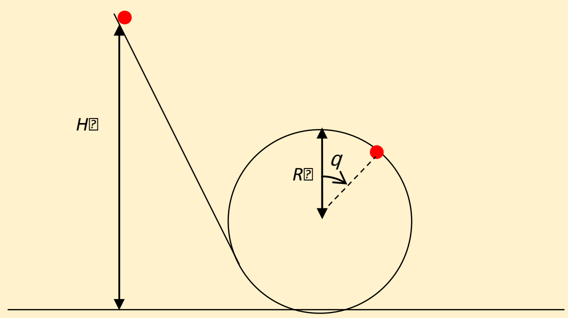
(a) Find the critical initial height $H_c$ below which the object cannot complete the whole journey along the circular track.
(a) 求初始高度的臨界值 $H_c$。若初始高度低於該臨界高度，則物體不能沿著圓形軌道完成整個行程。

(b) When $H < H_c$, the object will leave the track at some point. Find $H$ so that the object will hit the center of the circle after leaving the track.
(b) 當 $H < H_c$ 時，物體將在某個點離開軌道。求 $H$，使得物體在離開軌道後將撞到圓心。



Solution:

(a) Measure the angular position of the object by the angle $\theta$ defined in the figure below.




Considering the centripetal force, we have

$$N + mg\cos\theta = \frac{mv^2}{R}$$

where $N$ is the normal reaction of the track.

The object will leave the track when $N = 0$. One can see that this is possible only when $-90^\circ \le \theta \le 90^\circ$.

By conservation of energy, the speed $v$ is determined by

$$\frac{1}{2}mv^2 + mgR(1+\cos\theta) = mgH\ .$$

Solving the two equations for $N$, we have

$$N = -mg\cos\theta + \frac{2mgH - 2mgR(1+\cos\theta)}{R} = \frac{2mgH}{R} - mg(3\cos\theta + 2)$$

Hence the angular position of the point the object loses contact is determined by

$$\frac{2mgH}{R} - mg(3\cos\theta + 2) = 0 \Rightarrow \cos\theta = \frac{2}{3}\left(\frac{H}{R} - 1\right),$$

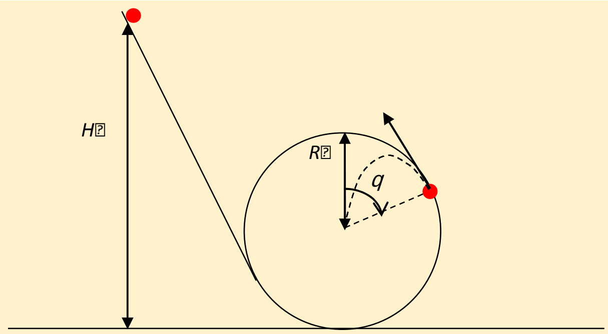
where $-90^\circ \le \theta \le 90^\circ \to \cos\theta \ge 0$.

One can see that there will be no solution when

$$\frac{2}{3}\left(\frac{H}{R} - 1\right) > 1 \Rightarrow H > \frac{5}{2}R.$$

Hence $H_c = 5R/2$. If $H < H_c$, the object will lose contact at some point.

(b) After the object leaves the track, it will perform projectile motion. By geometry, the launching angle is $\theta$.




Hence the trajectory is

$$y = R\left(1+\cos\theta\right) + x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta}.$$

The coordinate of the center is $\left(R\sin\theta, R\right)$.

If the center is on the trajectory

$$R = R\left(1+\cos\theta\right) + R\sin\theta\tan\theta - \frac{gR^2\sin^2\theta}{2v^2\cos^2\theta}$$

$$R\cos\theta + R\frac{\sin^2\theta}{\cos\theta} - \frac{gR^2\sin^2\theta}{2v^2\cos^2\theta} = 0$$

$$\cos^2\theta + \sin^2\theta - \frac{gR\sin^2\theta}{2v^2\cos\theta} = 0$$

$$gR\sin^2\theta = 2v^2\cos\theta$$

By energy conservation, $\frac{1}{2}mv^2 + mgR\left(1+\cos\theta\right) = mgH$ , we have

$$gR\sin^2\theta = 4\left[gH - gR\left(1+\cos\theta\right)\right]\cos\theta$$

$$R\left(1-\cos^2\theta\right) = 4\left[H - R\left(1+\cos\theta\right)\right]\cos\theta$$

$$R\left\{1-\left[\frac{2}{3}\left(\frac{H}{R}-1\right)\right]^2\right\} = 4\left[H - R\left(1+\frac{2}{3}\left(\frac{H}{R}-1\right)\right)\right]\frac{2}{3}\left(\frac{H}{R}-1\right)$$

$$12\left(H-R\right)^2 = 9R^2$$

$$H - R = \pm\frac{\sqrt{3}}{2}R$$

$$H = \left(1\pm\frac{\sqrt{3}}{2}\right)R$$

The solution $H = \left(1-\frac{\sqrt{3}}{2}\right)R < R$ is rejected because in this case $90° < \theta < 270°$ and the object will never leave the track.

Hence the solution is $H = \left(1+\frac{\sqrt{3}}{2}\right)R < \frac{5}{2}R$.