

Hong Kong Physics Olympiad 2020

2020 年香港物理奧林匹克競賽

Organisers 合辦機構

Education Bureau

教育局

The Hong Kong Academy for Gifted Education

香港資優教育學苑

The Hong Kong University of Science and Technology

香港科技大學

Advisory Organisations 顧問機構

The Physical Society of Hong Kong

香港物理學會

Hong Kong Physics Olympiad Committee

香港物理奧林匹克委員會

18 October, 2020

2020 年 10 月 18 日

Rules and Regulations 競賽規則

1. All questions are in bilingual versions. You can answer in either Chinese or English, but only ONE language should be used throughout the whole paper.

所有題目均為中英對照。你可選擇以中文或英文作答，惟全卷必須以單一語言作答。

2. On the first page of the answer books, please write your 3-digit Contestant Number and English Name.

在答題簿的第一頁上，請填上你的 3 位數字參賽者號碼及英文姓名。

3. The open-ended problems are quite long. Please read the whole problem first before attempting to solve them. If there are parts that you cannot solve, you are allowed to treat the answer as a known answer to solve the other parts.

開放式問答題較長，請將整題閱讀完後再著手解題。若某些部分不會做，也可把它們的答案當作已知來做其他部分。

The following symbols and constants are used throughout the examination paper unless otherwise specified:

除非特別注明，否則本卷將使用下列符號和常數：

Gravitational acceleration on Earth surface 地球表面重力加速度	g	9.80 ms^{-2}
Gravitational constant 重力常數	G	$6.67 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2}$

Multiple Choice Questions (2 Marks Each) 選擇題(每題 2 分)

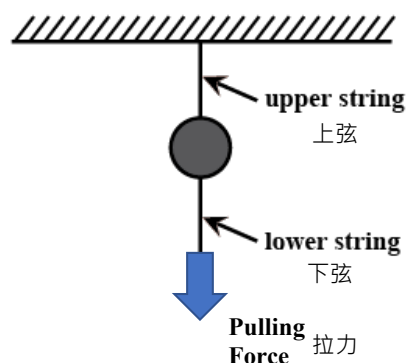
1. A 1.0-kg block and a 2.0-kg block are pressed together on a horizontal frictionless surface with a compressed light spring between them. They are not attached to the spring. After they are released and detached from the spring

1. 將 1.0 公斤重的物體和 2.0 公斤重的物體在無摩擦的水平表面上壓在一起，在它們之間有一個壓縮的輕彈簧，物體並沒有固定在彈簧上。當它們釋放並從彈簧離開之後

- A. both blocks will have the same amount of kinetic energy. 兩個物體具有相同的動能。
- B. both blocks will have equal speeds. 兩個物體具有相同的速率。
- C. the lighter block will have more kinetic energy than the heavier block. 較輕的物體比較重的物體具有更多動能。
- D. the magnitude of the momentum of the heavier block will be greater than the magnitude of the momentum of the lighter block. 較重的物體的動量將大於較輕的物體。
- E. the heavier block will have more kinetic energy than the lighter block. 較重的物體比較輕的物體具有更多動能。

2. A heavy ball is hung with a string from the ceiling. Another string is attached to the bottom of the ball. A downward pulling force is exerted on the lower string as shown in the figure below. Which of the following statements is true?

2. 用一根繩子從天花板上懸掛一個沉重的球。另一根繩子附在球的底部。一向下的拉力施加在下方的繩子上，如下圖所示。下列哪項是正確的？



- A. If the lower string is pulled suddenly by a large force, the upper string will break first. 如果一個巨大的拉力突然作用在下方的繩子上，則上方的繩子將首先斷裂。
- B. If the lower string is pulled suddenly by a large force, the lower string will break first. 如果一個巨大的拉力突然作用在下方的繩子上，則下方的繩子將首先斷裂。

- C. If the lower string is pulled suddenly by a large force, the two strings will break at the same time. 如果一個巨大的拉力突然作用在下方的繩子上，則兩條繩子將同時斷裂。
- D. If the lower string is pulled by a slowly increasing force, the lower string will break first. 如果一個緩慢增強的拉力作用在下方的繩子上，則下方的繩子將首先斷裂。
- E. If the lower string is pulled by a slowly increasing force, none of the string will eventually break. 如果一個緩慢增強的拉力作用在下方的繩子上，兩條繩子最終都不會中斷裂。

3. Which of the following statements is true in describing two satellites orbiting on circular orbits with the same radius around Earth?

3. 兩個以地球為圓心作相同半徑圓形繞行的人造衛星，以下哪句描述是正確的？

- A. The satellites could have different masses. 兩個人造衛星可能具有不同的質量。
- B. The satellites could have different periods. 兩個人造衛星可能具有不同的週期。
- C. The satellites could have different linear speeds. 兩個人造衛星可能具有不同的線性速率。
- D. The satellites could have different magnitudes of accelerations. 兩個人造衛星的加速度可能具有不同的大小。
- E. The gravitational acceleration measured inside the satellites could be different. 兩個人造衛星內觀測到的重力加速度可能不同。

4. A roller-coaster cart full of water is moving at a constant speed along a horizontal, frictionless length of track. Suddenly, a plug in the bottom of the cart is removed, and the water starts to flow downwards out of the cart. What happens to the speed of the cart while the water is flowing? Ignore air resistance in your answer.

4. 裝滿水的過山車以恆定的速率沿著無摩擦的水平軌道運動。突然，過山車底部的一個塞子打開，水開始向下流到過山車外面。當水在流動時，過山車的速率會如何變化？本問題可忽略空氣阻力。

- A. The cart speeds up. 過山車加速。
- B. The cart slows down. 過山車減速。
- C. The cart speeds up until half the water is gone, then it slows down. 過山車加速直到少一半的水，然後減速。

- D. The cart slows down until half the water is gone, then it speeds up. 過山車加減速直到少一半的水，然後加速。
- E. The cart's speed does not change. 過山車的速率不變。

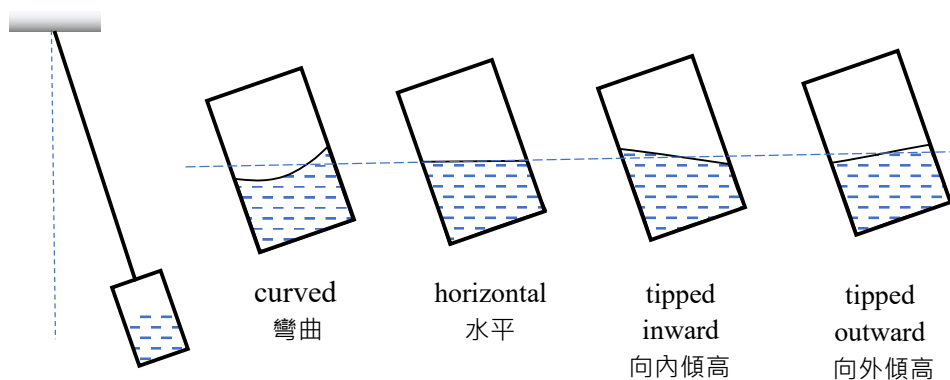
Solution: The water flowing out of the cart does not exert a horizontal force on the cart, so the cart's speed does not change. Alternatively, the water leaving the cart does carry horizontal momentum, so the momentum of the cart does decrease. However, the mass of the cart is decreasing by the same fraction, so the speed of the cart does not change.

Answer: E

5. A cylinder half-filled with water is hung from the ceiling by a long cable and swings with a small angle like a pendulum. At the position shown in the figure, the water level is observed to be (see the figure below for the description of the choices)

5. 半滿水的圓柱體通過長繩懸掛在天花板上，並像鐘擺一樣以小角度擺動。在圖中所示的位置，觀察到水位為（選擇說明請參見下圖）

- A. curved 彎曲
- B. horizontal 水平
- C. tipped inward 向內傾高
- D. tipped outward 向外傾高
- E. tipped outward when the pendulum is swinging outward, and tipped inward when the pendulum is swinging inward 鐘擺向外擺動時向外傾高，鐘擺向內擺動時向內傾高



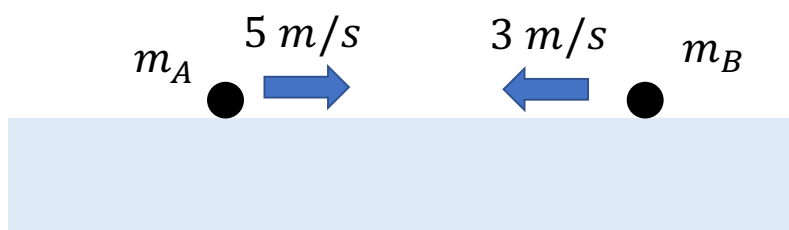
At the position shown in the figure, the acceleration due to the pendulum motion is dominated by the acceleration towards the central axis. This generates a fictitious force in the opposite direction (similar to the relation between centripetal force and centrifugal force in circular motion). The total force experienced by water is the sum of the downward gravitational force plus the rightward fictitious force. Since the cable is long, the water level is effectively in equilibrium, and the water surface is normal to the total force. Therefore, it should be tipped

outward. (In fact, the water surface is effectively orthogonal to the cable direction for small-angle oscillations, as can be verified by detailed analysis and experimental observation (see the video).)

Answer: D.

6. Consider the one-dimensional collision of two objects A and B with masses $m_A = 2 \text{ kg}$ and $m_B = 3 \text{ kg}$. The objects are made of unknown materials. A and B are initially moving with constant velocities of 5 m/s to the right and 3 m/s to the left, respectively, with A on the left side of B . Which of the following are possible final velocities of the objects after the collision?

6. 考慮質量為 $m_A = 2 \text{ kg}$ 和 $m_B = 3 \text{ kg}$ 的兩個物體 A 和 B 的一維碰撞。這些物體由未知材料製成，可以視為質點。 A 和 B 最初分別以向右 5 m/s 和向左 3 m/s 的恆定速度移動，而 A 在 B 的左側。以下哪項是物體在物體經過碰撞之後可能的最終速度？



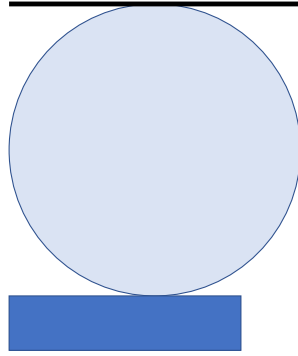
- | | | |
|-----|--|---------------------------------|
| I | A: 3 m/s to the left, B: 2 m/s to the right | A : 向左 3 m/s · B : 向右 2 m/s |
| II | A: 2.5 m/s to the left, B: 2 m/s to the right | A : 向左 2.5 m/s · B : 向右 2 m/s |
| III | A: 4.6 m/s to the left, B: 3.4 m/s to the right. | A : 向左 4.6 m/s · B : 向右 3.4 m/s |
| IV | A: 0.2 m/s to the right, B: 0.2 m/s to the right | A : 向右 0.2 m/s · B : 向右 0.2 m/s |

- | | | |
|------------------------|-------------------------|--------------------|
| A. II only | B. III only | C. II and III only |
| D. I, II, and III only | E. II, III, and IV only | |

7. A right-angled plate with equal length on both sides is placed horizontally on a fixed cylinder. The plate has uniform density and the length of each side is $2R$, where R is the radius of the cylinder. What is the minimum coefficient of static friction μ between the cylinder and the plate to prevent the plate from slipping off?

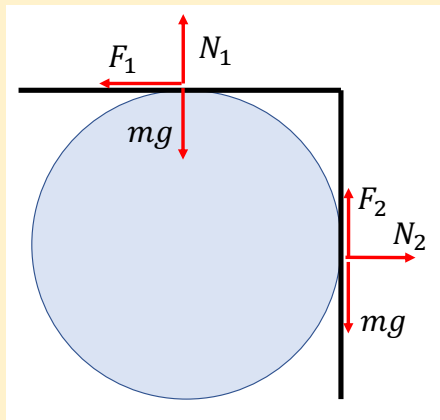
7. 一塊兩邊長度相等的直角板水平地放置在固定的圓柱體上。直角板的密度均勻，每邊長度為 $2R$ ，其中 R 為圓柱體的半徑。求圓柱體和板之間的靜摩擦系數 μ 的最小值，才能使板子不滑落？

- | | | | | |
|----------|----------|----------|----------|----------|
| A. 0.414 | B. 0.207 | C. 0.104 | D. 0.828 | E. 0.312 |
|----------|----------|----------|----------|----------|



Solution:

The free body diagram of the L-shaped plate is shown.



The equilibrium of forces and torques give

$$\begin{aligned} N_1 + F_2 &= 2mg \\ F_1 &= N_2 \\ (mg + N_2)R &= N_1R \end{aligned}$$

The frictional forces must satisfy

$$\begin{aligned} F_1 &\leq \mu N_1 \\ F_2 &\leq \mu N_2 \end{aligned}$$

We conclude that

$$\begin{aligned} F_1 + F_2 &= mg. \quad [1] \\ F_1 &\leq \mu N_1 = \mu(mg + N_2) = \mu(mg + F_1) \Rightarrow F_1 \leq \frac{\mu}{1 - \mu} mg \\ F_2 &\leq \mu F_1 = \frac{\mu^2}{1 - \mu} mg \end{aligned}$$

Sub. Into equation [1],

$$\begin{aligned} mg &\leq \left(\frac{\mu}{1 - \mu} + \frac{\mu^2}{1 - \mu} \right) mg \\ &\Rightarrow \mu^2 + \mu \geq 1 - \mu \\ &\Rightarrow \mu^2 + 2\mu - 1 \geq 0 \\ &\Rightarrow (\mu - \mu_+)(\mu - \mu_-) \geq 0 \end{aligned}$$

where

$$\mu_{\pm} = \frac{-2 \pm \sqrt{4 + 4}}{2} = -1 \pm \sqrt{2}$$

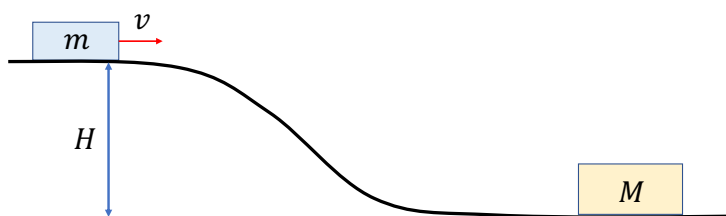
since $\mu > 0 > \mu_-$, we must conclude that

$$\Rightarrow \mu \geq \mu_+ = \sqrt{2} - 1 \approx 0.414$$

Answer: A.

8. A trolley with mass m moves along the horizontal track at speed v . The track continues to extend downward and smoothly connects to a new horizontal track at height H below. There is a stationary carriage with mass M here ($M > m$). The trolley moves towards the carriage, causing a completely elastic collision. What is the minimum initial speed v of the trolley such that it can return to the upper horizontal section? The friction can be neglected.

8. 質量為 m 的小車以速度 v 沿水平放置的軌道運動，軌道繼續向下延伸並平坦過渡到高度降低了 H 的新水平軌道。此處有靜止的小車廂，質量為 M ($M > m$)，小車滑向小車廂，發生完全彈性碰撞。求小車初速度 v 的最小值，使它可以返回到上部水平路段。摩擦不計。



A. $\frac{2\sqrt{2MmgH}}{M-m}$

B. $\frac{\sqrt{2MmgH}}{M-m}$

C. $2\sqrt{2mgH}$

D. $2\sqrt{2MgH}$

E. $\sqrt{2mgH}$

Solution:

The collision velocity of the trolley before the collision is

$$v_1 = \sqrt{v^2 + 2gH}$$

By the conservation of mechanical energy and momentum, we have

$$\begin{aligned} mv_1 &= mv'_1 + Mv'_2 \\ \frac{1}{2}mv_1^2 &= \frac{1}{2}mv_1'^2 + \frac{1}{2}Mv_2'^2 \\ \Rightarrow v_1 + v_1' &= v_2' \\ \Rightarrow v_1' &= \frac{m-M}{m+M}v_1 < 0 \end{aligned}$$

The trolley rebounds after the collision. In order to return to the original height, we have

$$\begin{aligned} \frac{1}{2}mv_1'^2 &> mgH \\ \Rightarrow \left(\frac{M-m}{M+m}\right)^2 v_1^2 &> 2gH \\ \Rightarrow \left(\frac{M-m}{M+m}\right)^2 (v^2 + 2gH) &> 2gH \\ \Rightarrow v^2 &> \left(\left(\frac{M+m}{M-m}\right)^2 - 1\right) 2gH = \frac{8gHMm}{(M-m)^2} \end{aligned}$$

$$\Rightarrow v > \frac{2\sqrt{2MmgH}}{M-m}$$

Answer: A.

9. Two highways intersect at right angles as shown in the figure. At the instant shown, car A is located at $d = 5$ km from the intersection and is traveling at speed $v_A = 80$ km/h. Car B is located at the intersection and is traveling at speed $v_B = 60$ km/h. When the two cars are closest to each other, car B has travelled a distance of

9. 如圖所示，兩條高速公路成直角相交。在所示的瞬間，汽車 A 距交叉路口 $d = 5$ km，行駛速度為 $v_A = 80$ km/h。汽車 B 位於交叉路口，行駛速度為 $v_B = 60$ km/h。當兩輛汽車彼此最接近時，汽車 B 行駛了的距離是

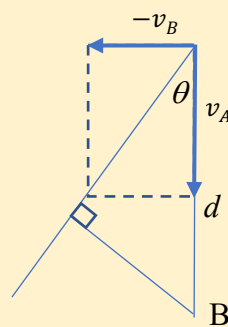
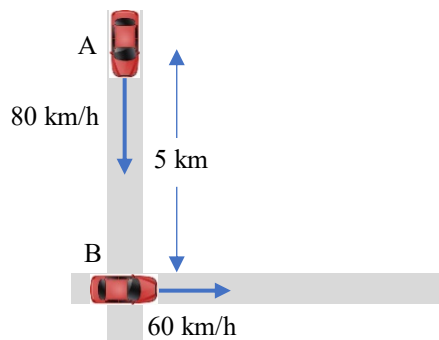
A. 1.8 km

B. 2.4 km

C. 3 km

D. 3.2 km

E. 4 km



Consider the velocity of car A relative to car B.

$$v_{A|B} = \sqrt{80^2 + 60^2} = 100 \text{ km/h.}$$

When the two cars are closest, distance travelled by car A in the reference frame of car B

$$d_{A|B} = d \cos \theta = d \frac{v_A}{v_{A|B}} = 5 \left(\frac{80}{100} \right) = 4 \text{ km.}$$

Time elapsed:

$$t = \frac{d_{A|B}}{v_{A|B}} = \frac{4}{100} = 0.04 \text{ h.}$$

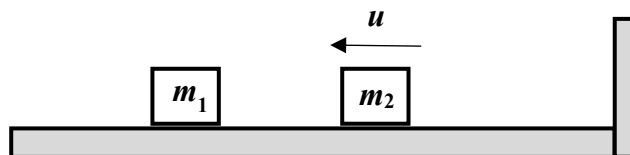
Distance traveled by car B:

$$d_B = v_B t = (60)(0.04) = 2.4 \text{ km.}$$

Answer: B.

10. As shown in the figure, a block of mass m_1 is at rest on a long frictionless table that is touching a wall. Block 2 of mass m_2 is placed between block 1 and the wall and is moving to the left at speed u . After block 2 has collided once with block 1 and once with the wall, both blocks move with the same velocity. Assume all collisions are elastic. The value of m_2 is

10. 如圖所示，質量 m_1 的物塊停在長的無摩擦工作台上，該工作台接觸牆壁。質量為 m_2 的物塊 2 位於物塊 1 和牆之間，並以速度 u 向左移動。在物塊 2 與物塊 1 發生碰撞而物塊 2 再與牆發生碰撞後，兩個物塊以相同的速度移動。假設所有碰撞都是彈性的。 m_2 的值為



A. $m_1/3$

B. $m_1/2$

C. m_1

D. $2m_1$

E. $3m_1$

Conservation of linear momentum:

$$m_2 u = m_2 v_2 + m_1 v_1. \quad (1)$$

Conservation of energy:

$$\frac{1}{2} m_2 u^2 = \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_1 v_1^2. \quad (2)$$

From (1),

$$m_2(u - v_2) = m_1 v_1.$$

From (2),

$$m_2(u^2 - v_2^2) = m_1 v_1^2.$$

Dividing the two equations,

$$u + v_2 = v_1. \quad (3)$$

From (1) and (3),

$$v_1 = \frac{2m_2 u}{m_1 + m_2},$$
$$v_2 = \frac{(m_2 - m_1)u}{m_1 + m_2}.$$

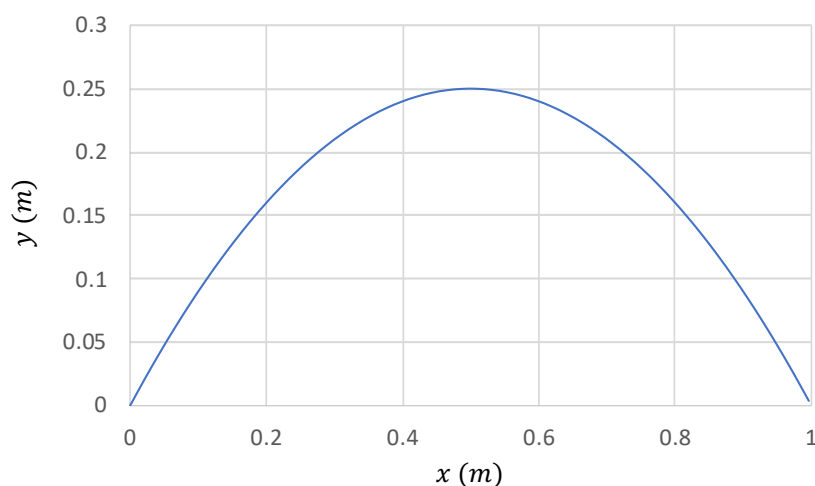
After block 2 has collided with the wall, its velocity becomes $-v_2$. Hence

$$v_1 = -v_2,$$
$$m_2 = \frac{m_1}{3}.$$

Answer: A.

11. The trajectory of a toy cannon is shown in the figure below. Assuming that the air resistance during the flight can be neglected, what is the launching angle θ of the projectile?

11. 一個玩具砲彈的飛行軌跡如下圖所示。假設飛行中的空氣阻力可以忽略不計，請問砲彈的發射仰角 θ 是多少？



- A. 15° B. 30° C. 45° D. 60° E. 75°

Answer: C.

12. Following the previous question, what is the initial speed of the cannon?

12. 接上題，請問砲彈的初速是多少？

- A. 9.8 m/s B. 6.26 m/s C. 4.43 m/s
D. 3.36 m/s E. 3.13 m/s

Solution for Q11 and Q12:

The trajectory of the parabolic is

$$\begin{aligned}x &= v \cos \theta t \\y &= v \sin \theta t - \frac{1}{2}gt^2 \\ \Rightarrow y &= \tan \theta x - \frac{1}{2}g \frac{x^2}{v^2 \cos^2 \theta}\end{aligned}$$

The range of the projectile is given by

$$R = \frac{2 \sin \theta \cos \theta}{g} v^2 \quad [1]$$

And the maximum height is

$$H = \frac{v^2 \sin^2 \theta}{2g}$$

$$\Rightarrow \frac{R}{H} = \frac{4 \sin \theta \cos \theta}{\sin^2 \theta} = \frac{4}{\tan \theta}$$

$$\Rightarrow \tan \theta = \frac{4H}{R}$$

From the graph, we have $H = 0.25m$ and $R = 1m$

$$\Rightarrow \tan \theta = 1$$

$$\Rightarrow \theta = 45^\circ$$

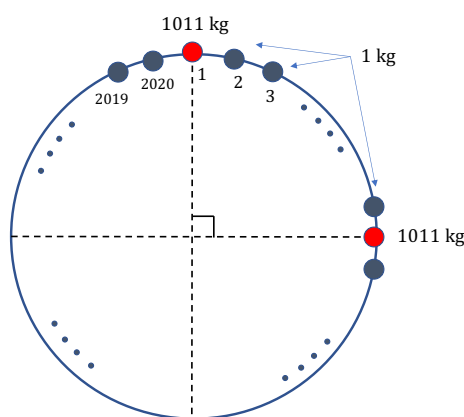
Substitute into Eqtn [1],

$$1 = \frac{v^2}{g} \Rightarrow v = \sqrt{g} = 3.13 \text{ m/s}$$

Answer: E.

13. 2020 point particles are equally spaced on a circle with radius 1 m. The particles are identical with mass 1 kg except two, both with mass 1011 kg. The angle between the radii joining these two particles to the center is 90° . What is the distance (in m) of the center of mass of the system from the center of the circle?

13. 2020 個質點在半徑 1 m 的圓上等距分佈。除了兩個質點質量各為 1011 kg 外，其餘每個質點質量均為 1 kg。該兩個較重質點連接到中心的半徑之間的角度為 90° 。問系統質心距圓心的距離（以 m 為單位）是多少？



A. $\frac{1}{4}$

B. $\frac{1}{2\sqrt{2}}$

C. $\frac{1011}{2020\sqrt{2}}$

D. $\frac{1}{2}$

E. $\frac{1}{\sqrt{2}}$

Answer:

For simplicity put the circle's center at the origin with the two heavier masses having radii making angle 45° and -45° with the positive x-axis, so that the system is symmetric about the x-axis.

Consider it as the composite system of 2020 point particles all with mass 1 kg, and two particles with mass 1010 kg. The CM of the 2020 identical particles is obviously at the origin. So

$$x_{CM} = \frac{2020 \times 0 + 1010 \times \cos 45^\circ + 1010 \times \cos 45^\circ}{2020 + 1010 + 1010} = \frac{1010\sqrt{2}}{4040} = \frac{1}{2\sqrt{2}}$$

By symmetry,

$$y_{CM} = 0$$

Answer: B.

14. A truck is travelling on a straight road under a constant acceleration of 4.0 ms^{-2} . A simple pendulum hung from the ceiling of the truck performs small angle oscillations. The length of the pendulum is 50 cm and the mass of the bob is 2.0 kg. What is the frequency f of the oscillations?

14. 卡車以 4.0 ms^{-2} 的恆定加速度在直路上行駛。懸掛在卡車天花板上的簡單擺錘以小角度振動。擺錘的長度是 50 cm。擺錘質量是 2.0 kg。問振蕩的頻率 f 是多少？

A. 0.70 Hz B. 0.73 Hz C. 0.84 Hz D. 4.4 Hz E. 4.6 Hz

Ans: The effective gravity has a vertically downward component of g and a horizontal component of 4.0 m/s^2 opposite to the direction of acceleration. The magnitude is hence

$$g_{\text{eff}} = \sqrt{9.8^2 + 4.0^2} = 10.6$$

The frequency is

$$f = \frac{1}{2\pi} \sqrt{\frac{g_{\text{eff}}}{L}} = \frac{1}{2\pi} \sqrt{\frac{10.6}{0.50}} = 0.73 \text{ Hz}$$

Answer: B.

15. Following the previous question. When the pendulum is at equilibrium, what is its angle from the vertical direction?

15. 承上題。當擺錘處於平衡狀態時，它與垂直方向的夾角是多少？

A. 0° B. 14° C. 22° D. 35° E. 43°

Ans:

In equilibrium, the pendulum should be along the direction of \vec{g}_{eff} . Hence the angle should be

$$\tan^{-1} \frac{4.0}{9.8} = 22^\circ$$

Answer: C.

16. Consider two concentric uniform thin shells. The inner shell has radius R_1 and mass M_1 . The outer shell has radius R_2 and mass M_2 . A point particle with mass m is initially at infinity and moves to a final distance of r from the center, where $R_1 < r < R_2$ (assuming that the particle can penetrate a shell). What is the work done by the gravitational force of the shells in the above motion?

16. 考慮兩個同心的均勻薄球殼。內殼的半徑為 R_1 、質量為 M_1 。外殼的半徑為 R_2 、質量為 M_2 。質量為 m 的質點最初位於無窮遠處，並且移動到與圓心距離為 r 的最終位置，其中 $R_1 < r < R_2$ (假設質點可以穿透殼)。求在上述運動中兩個薄球殼的重力所作的功。

A. $\frac{GM_1 m}{r}$

B. $-\frac{GM_1 m}{r}$

C. $\frac{GM_1 m}{r} + \frac{GM_2 m}{r}$

D. $\frac{GM_1 m}{r} + \frac{GM_2 m}{R_2}$

E. $-\frac{GM_1 m}{r} - \frac{GM_2 m}{R_2}$

Ans:

The potential energy due to the inner shell is

$$-\frac{GM_1 m}{r}$$

By definition of PE, the work-done by the gravity of the inner shell is

$$\frac{GM_1 m}{r}$$

The potential energy due to the outer shell when the mass moves from infinity to its surface is

$$-\frac{GM_2 m}{R_2}$$

By definition of PE, the work done by the gravity of the outer shell in moving the mass from infinity to its surface is

$$\frac{GM_2 m}{R_2}$$

Inside the outer shell, there is no gravitational force due to the outer shell.

Therefore, the work done by the gravity of the outer shell in moving the mass from its surface to r is 0.

Hence the total work done by the outer shell is

$$\frac{GM_2 m}{R_2}$$

By superposition principle, the work done by the gravity of the two shells is

$$\frac{GM_1m}{r} + \frac{GM_2m}{R_2}$$

Answer: D.

17. The equation

$$x = \frac{1}{x^2} - \frac{1}{\alpha(1-x)^2}$$

is a quintic equation in x which has no analytic solution in general.

The numerical (real) solution is approximately 0.849 when $\alpha = 81.3$, and is 0.892 when $\alpha = 238$, and is 0.920 when $\alpha = 597$.

The mass of a moon is 7.35×10^{22} kg. The mass of the parent planet is 1.75×10^{25} kg so that when considering their orbital motion one can consider the planet to be at rest. Both the planet and the moon can be considered as point objects. One may also assume that the moon's orbital motion is perfectly circular with a radius of 3.00×10^5 km. An artificial satellite can stay on the line between the planet and the moon at fixed distances from them. What approximately is the distance of this satellite from the planet? Choose the best answer.

17.

$$x = \frac{1}{x^2} - \frac{1}{\alpha(1-x)^2}$$

是 x 的一個五次方程式，沒有一般解析解。當 $\alpha = 81.3$ 時，數值（實）解的解近似為 0.849，當 $\alpha = 238$ 時為 0.892，當 $\alpha = 597$ 時為 0.920。

一個衛星的質量為 7.35×10^{22} kg。母行星的質量為 1.75×10^{25} kg，因此在考慮它們的軌道運動時可以認為行星處於靜止狀態。行星和衛星都可以視為質點。可假設衛星的軌道運動是完美的圓形，半徑為 3.00×10^5 km。有一顆人造衛星可以處於行星和衛星之間的直線上，它與行星和衛星的距離保持不變。這顆人造衛星與行星的距離是多少？選擇最佳答案。

A. 32,400 km

B. 45,300 km

C. 255,000 km

D. 268,000 km

E. 276,000 km

Ans:

Let the mass of the planet, the moon, and the artificial satellite be m_P , m_M , and m_S , respectively.

Consider the orbital motion of the moon, we have

$$m_M \omega^2 R = \frac{Gm_P m_M}{R^2}$$

Hence the angular speed is

$$\omega = \sqrt{\frac{Gm_p}{R^3}}$$

The mass ratio is

$$\frac{m_p}{m_M} = 238$$

Let the distance of the artificial satellite from the planet be r , then

$$\begin{aligned} m_S \omega^2 r &= \frac{Gm_p m_S}{r^2} - \frac{Gm_M m_S}{(R-r)^2} \\ \frac{Gm_p}{R^3} r &= \frac{Gm_p}{r^2} - \frac{Gm_M}{(R-r)^2} \\ m_p \frac{r}{R} &= \frac{m_p}{(r/R)^2} - \frac{m_M}{(1-r/R)^2} \\ \frac{r}{R} &= \frac{1}{(r/R)^2} - \frac{1}{(m_p/m_M)(1-r/R)^2} \end{aligned}$$

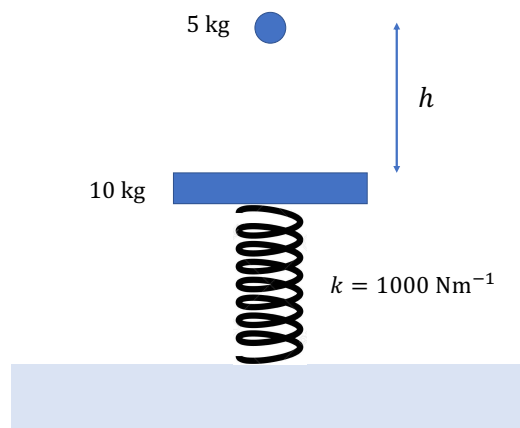
Therefore,

$$\begin{aligned} \frac{r}{R} &\approx 0.892 \\ r &\approx 0.892 \times 3.00 \times 10^5 \text{ km} \approx 268000 \text{ km} \end{aligned}$$

Answer: D.

18. An object of mass 5.00 kg is initially at a height of h above a 10.0 kg-flat plate. The plate is supported by a long spring below with spring constant 1000 Nm^{-1} , which is initially at equilibrium under gravity and the spring force. The object is then released from rest to hit the plate in a perfectly inelastic collision. Find the maximum height h below which the object will always remain in contact with the plate in the subsequent oscillations.

18. 質量為 5.00 kg 的物體最初位於 10.0 kg 的平板上方的高度 h 處。平板由下方的長彈簧支撐，彈簧常數為 1000 Nm^{-1} 。在初始時，平板在重力和彈簧力的作用下處於平衡狀態。現在將物體從靜止狀態釋放以撞擊平板，撞擊過程為完全非彈性碰撞。求出最大高度 h ，以使在隨後的振盪中，物體始終保持與平板接觸。



- A. 20 cm
D. 59 cm

- B. 22 cm
E. 66 cm

- C. 45 cm

Ans:

Let the mass of the object and the plate be m and M , respectively.

By energy conservation, the speed of the object just before the collision is

$$\frac{1}{2}mu^2 = mgh \Rightarrow u = \sqrt{2gh}$$

By momentum conservation, the common speed after collision is

$$mu = (m + M)v \Rightarrow v = \frac{m}{m + M}u = \frac{m}{m + M}\sqrt{2gh}$$

The new equilibrium position is lowered by $\frac{mg}{k}$.

The amplitude of the subsequent SHM can be obtained by

$$\frac{1}{2}kA^2 = \frac{1}{2}(m + M)v^2 + \frac{1}{2}k\left(\frac{mg}{k}\right)^2$$
$$A = \sqrt{\frac{m + M}{k}v^2 + \left(\frac{mg}{k}\right)^2}$$

The maximum magnitude of acceleration is

$$\frac{k}{m + M}A = \sqrt{\frac{k}{m + M}v^2 + \left(\frac{mg}{m + M}\right)^2} = \sqrt{\frac{2m^2gkh}{(m + M)^3} + \left(\frac{mg}{m + M}\right)^2}$$

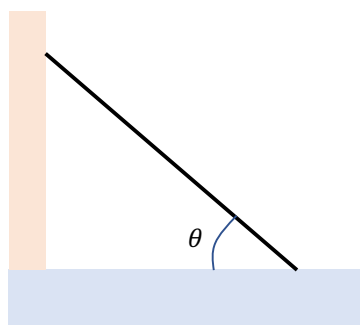
Set this to be smaller than or equal to g , we have

$$g^2 \geq \frac{2m^2gkh}{(m + M)^3} + \left(\frac{mg}{m + M}\right)^2$$
$$h \leq \frac{M(m + M)(2m + M)g}{2m^2k} = 59\text{cm}$$

Answer: D.

19. A 2.0-m long ladder leans against a smooth wall, making an angle of θ with the rough floor. The mass of the ladder is 10 kg and the linear mass density along its length is constant. The coefficient of static friction between the ladder and the floor is 0.20. When the angle θ is smaller than a certain critical value, the ladder cannot be in equilibrium. What is this critical angle?

19. 一個 2.0 m 長的梯子靠在光滑的牆壁上，與粗糙的地板成角度 θ 。梯子的質量為 10 kg，沿其長度的線性質量密度為常數。梯子與地板之間的靜摩擦係數為 0.20。當角度 θ 小於某個臨界值時，梯子不能保持平衡。這個臨界角是多少？



A. 11°

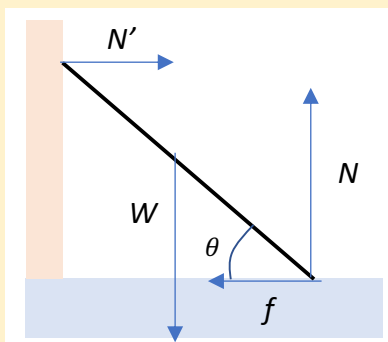
B. 22°

C. 66°

D. 68°

E. 79°

Ans:



$$f = N'$$

$$N = W$$

Take moment about contact point,

$$W \frac{L}{2} \cos \theta = N' L \sin \theta = f L \sin \theta$$

$$f = \frac{W}{2} \cot \theta = \frac{N}{2} \cot \theta \leq \mu_s N$$

$$\theta \geq \cot^{-1}(2\mu_s) = \cot^{-1}(0.40) = 68^\circ$$

Answer: D.

20. The tip of iceberg has a volume of $1.5 \times 10^9 \text{ m}^3$ above the sea level. Given the density for the iceberg is 920 kgm^{-3} and the density for sea water is 1030 kgm^{-3} . What is the volume of the iceberg submerged in water?

20. 冰山浮在海面上的一角的體積為 $1.5 \times 10^9 \text{ m}^3$ 。已知冰山的密度為 920 kgm^{-3} ，海水的密度為 1030 kgm^{-3} 。冰山在水中的體積是多少？



A. $14.05 \times 10^9 \text{ m}^3$

B. $12.55 \times 10^9 \text{ m}^3$

C. $1.68 \times 10^9 \text{ m}^3$

D. $1.50 \times 10^9 \text{ m}^3$

E. $0.18 \times 10^9 \text{ m}^3$

Solution:

Let V_1 and V_2 be the volume of the iceberg above and below sea level.

Balance of force:

$$\rho_{\text{water}} V_2 g = \rho_{\text{ice}} (V_1 + V_2) g$$

We have

$$\begin{aligned} V_2 &= V_1 \frac{\rho_{\text{ice}}}{\rho_{\text{water}} - \rho_{\text{ice}}} \\ &= 1.5 \times 10^9 \text{ m}^3 \times \frac{920}{1030 - 920} \\ &= 1.255 \times 10^9 \text{ m}^3 \end{aligned}$$

Answer: B.

21. A basketball is dropped vertically at a height of 1.5m. Initially, it is at rest and if every time it bounces back from the ground, it loses 30% of its energy. How long can the basketball stay in motion until it becomes completely at rest again?

籃球在 1.5m 的高度垂直下落。最初，它處於靜止狀態，如果每次從地面反彈回來，它都會損失 30% 的能量。籃球可以運動多長時間，直到完全恢復靜止？

A. 1.84 s

B. 3.14 s

C. 3.39 s

D. 6.22 s

E. 6.78 s

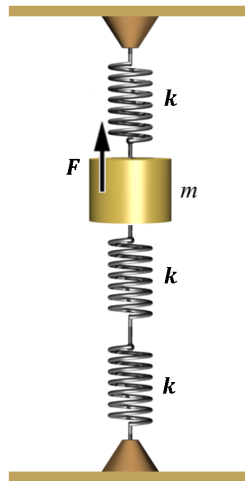
Solution:

When the ball is released, it takes time $T = \sqrt{2h/g}$. After a bounce, the height it can achieve is a factor of $s = 0.7$ of the height in the previous cycle. Therefore, the time it requires become coming to rest again is

$$\begin{aligned} &\sqrt{\frac{2h}{g}} + 2\sqrt{\frac{2sh}{g}} + 2\sqrt{\frac{2s^2h}{g}} + 2\sqrt{\frac{2s^3h}{g}} + \dots \\ &= \sqrt{\frac{2h}{g}} \left(\frac{1 + \sqrt{s}}{1 - \sqrt{s}} \right) \\ &= \sqrt{2 \times \frac{1.5}{9.8} \frac{1 + \sqrt{0.7}}{1 - \sqrt{0.7}}} = 6.22 \text{ s} \end{aligned}$$

22. A mass m is connected to 3 identical springs with spring constant k , as shown in the figure. The whole set is then mounted to a fixed ceiling and floor and is allowed to settle to equilibrium. Now, we apply a force F in the upward direction to displace the mass. What will be the displacement from its equilibrium position?

22. 如圖所示，質量 m 的物塊連接到三個彈簧常數均為 k 的相同彈簧上。整個系統固定在天花板 and 地板上，並達到平衡。現在，我們施加一個向上的力 F 來移動物塊。物塊相對平衡位置的位移是多少？



A. $2F/(3k)$

B. F/k

C. $(F - mg)/(3k)$

D. $(F + mg)/(3k)$

E. $F/(3k)$

Solution:

For the mass displaced by Δy in the positive vertical direction, it causes a further shrinkage of Δy for the upper spring, making a downward force of $k\Delta y$. It also causes a further stretching of $\Delta y/2$ for either one of the lower springs. It causes a force in the downward direction of $k\Delta y/2$. Therefore, the total downward force is

$$F = k\Delta y + k\Delta y/2 = 3k \Delta y/2$$

Equivalently, for a given force F , the displacement of the mass is

$$2F/(3k)$$

23. A block of mass m is hung from the system of massless springs with force constants shown in the figure. The downward extension of the spring system caused by the block is

23. 質量 m 的物塊懸掛在無質量的彈簧系統上，各彈簧的彈簧常數如圖所示。彈簧系統由物塊導致的向下延伸是

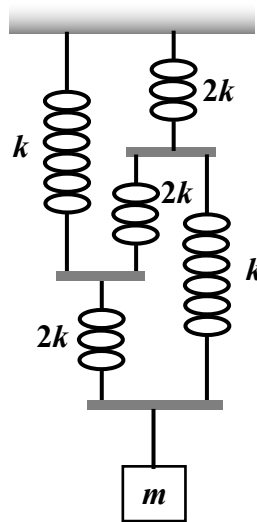
A. $\frac{7}{10} \frac{mg}{k}$

B. $\frac{22}{31} \frac{mg}{k}$

C. $\frac{3}{4} \frac{mg}{k}$

D. $\frac{17}{15} \frac{mg}{k}$

E. $\frac{3}{2} \frac{mg}{k}$



Let x_1, x_2, x_3 be the extended positions of the lower, middle and upper platforms with respect to the ceiling. Then we have

$$mg = 2k(x_1 - x_2) + k(x_1 - x_3), \quad (1)$$

$$2k(x_1 - x_2) = kx_2 + 2k(x_2 - x_3), \quad (2)$$

$$k(x_1 - x_3) + 2k(x_2 - x_3) = 2kx_3. \quad (3)$$

From (2) and (3),

$$5x_2 - 2x_3 = 2x_1,$$

$$5x_3 - 2x_2 = x_1.$$

The solution is $x_2 = \frac{4}{7}x_1$ and $x_3 = \frac{3}{7}x_1$. Substituting into (1),

$$x_1 = \frac{7}{10} \frac{mg}{k}.$$

Answer: A.

24. As shown in Fig. 1, a massive block is driven by a motor to exercise vertical motion. The acceleration due to the periodic motion is much higher than the gravitational acceleration. The displacement of the block versus time for two cycles is shown in Fig. 2. (The units of the vertical axis are arbitrary.)

24. 如圖 1 所示，具質量的物塊由電動機驅動以進行垂直運動。週期性運動引起的加速度遠高於重力加速度。圖 2 中顯示了兩個週期內物塊的位移隨時間的變化。（垂直軸的單位是任意的。）

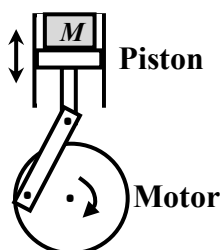


Fig. 1

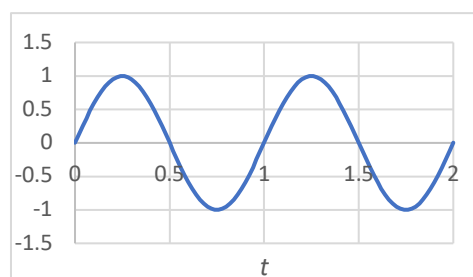
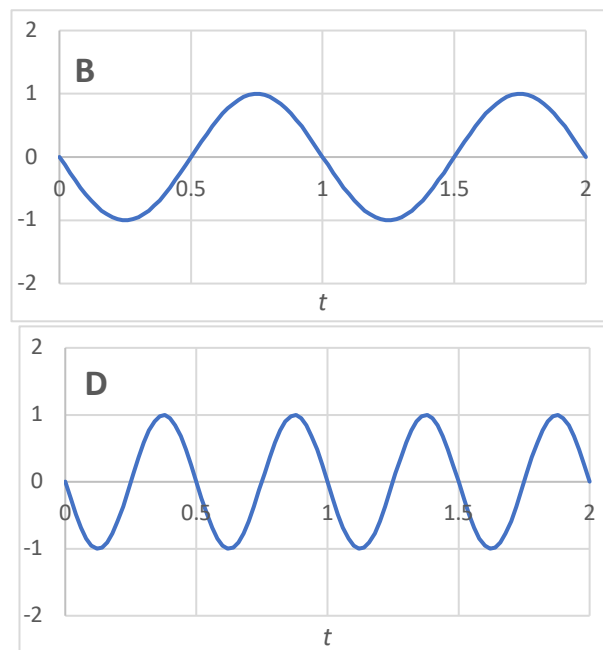
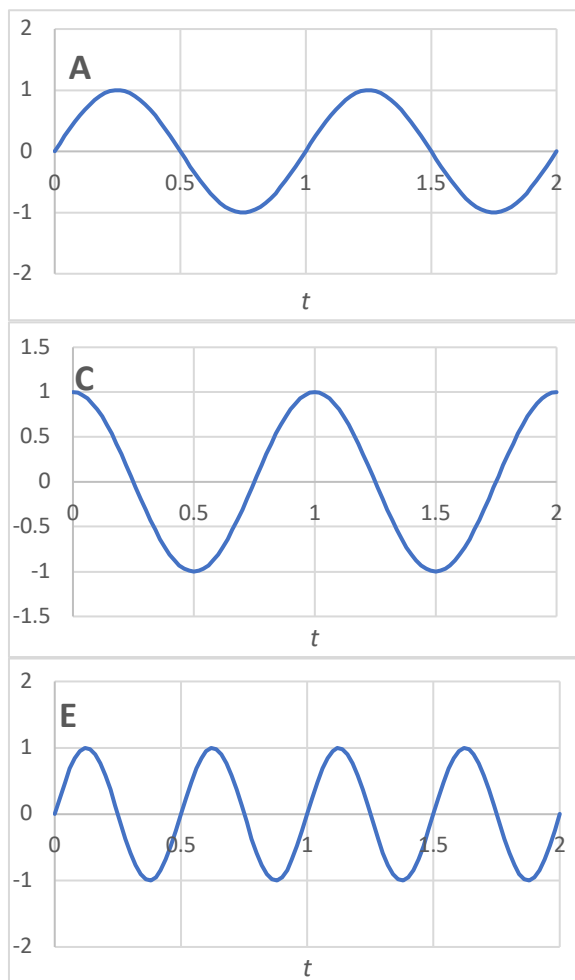


Fig. 2

Which of the following plots correspond to the power of the motor?

以下哪條曲線對應於電動機的功率？

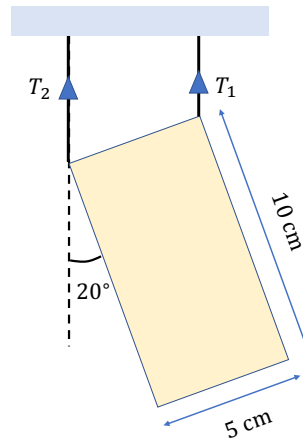


$$\begin{aligned}
 x &= A \sin \omega t, \\
 v &= \omega A \cos \omega t, \\
 F &= ma = -m\omega^2 A \sin \omega t. \\
 P &= Fv = -\omega^3 A^2 \sin \omega t \cos \omega t = -\omega^3 A^2 \sin(2\omega t) / 2.
 \end{aligned}$$

Answer: D.

25. A uniform rectangular slab of width 5 cm, height 10 cm and negligible thickness is hung from the ceiling by two rods at the upper edges. When the slab is tilted by 20° , the ratio of the tensions T_1/T_2 is equal to

25. 寬 5 cm，高 10 cm，厚度可忽略不計的均勻矩形板，用兩根位於上方的桿將其懸掛在天花板上。當矩形板傾斜 20° 時，張力 T_1/T_2 的比是



A. -2.14

B. 2.14

C. 0.15

D. 6.35

E. -6.35

The horizontal displacement of the center of mass of the slab from the left rod is $5 \sin 20^\circ + 2.5 \cos 20^\circ$.

The horizontal displacement of the center of mass of the slab from the right rod is $5 \sin 20^\circ - 2.5 \cos 20^\circ$.

At equilibrium, we have

$$T_2(5 \sin 20^\circ + 2.5 \cos 20^\circ) + T_1(\sin 20^\circ - 2.5 \cos 20^\circ) = 0$$

$$\frac{T_1}{T_2} = \frac{5 \sin 20^\circ + 2.5 \cos 20^\circ}{-5 \sin 20^\circ + 2.5 \cos 20^\circ} = 6.35$$

Please notice that at $\theta = 0$, the ratio $\frac{T_1}{T_2}$ should be equal to +1.

Answer: D.

Open Problems 開放題

The information below may be useful for solving the problems.

以下資料對於解題可能有幫助。

A. Kinetic friction 動摩擦

A body moving on a surface experiences a frictional force in the opposite direction of its movement. The magnitude of the frictional force is $f = \mu N$, where μ is called the coefficient of kinetic friction μ and N is the normal force acting on the body.

當一個物體在平面上運行時，它會受到一個與其運動方向相反的摩擦力。摩擦力的大小為 $f = \mu N$ ，其中 μ 稱為動摩擦係數， N 是作用在物體上的法向力。

B. Elastic potential energy 彈性勢能

The potential energy stored in a compressed spring is given by $E = \frac{1}{2}k(\Delta x)^2$, where k is the spring constant and Δx is the compressed length of the spring.

存儲在壓縮彈簧中的勢能是 $E = \frac{1}{2}k(\Delta x)^2$ ，其中 k 是彈簧常數， Δx 是彈簧的壓縮長度。

1. (15 points) As shown in the figure below, a uniform rectangular block of mass M is resting on a smooth horizontal surface. A small cube with mass m and negligible length and width is on the top of the rectangular block. At time $t = 0$, the cube moves from the left end of the block to the right with an initial velocity v . It finally stops at half the length of the rectangular block. It is known that the coefficient of kinetic friction between the small cube and the rectangular block is μ , and the gravitational acceleration is g .

(15 分) 如下圖所示，在一光滑水平面上，靜置有一質量為 M 的均勻長方體。一質量為 m ，長與寬均可忽略的小方塊，在此長方體的上表面，於時間 $t = 0$ 時，由左端以水平初速 v 開始向右滑行，最後停止在長方體的一半長度處。已知小方塊與長方體之間的動摩擦係數為 μ ，重力加速度為 g 。

- (a) (4 points) What is the time τ for the small cube to stop on the rectangular block?

(4 分) 求小方塊滑行至停止在長方體上的時間 τ ？

- (b) (4 points) From time $t = 0$ to τ , what are the distances travelled by the small cube and the rectangular block respectively?

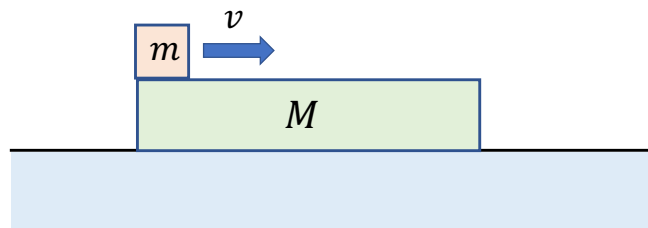
(4 分) 在 $t = 0$ 至 τ 期間，小方塊和長方體滑行的距離分別是多少？

(c) (4 points) What is the length L of the rectangular block? (Express your answer in terms of v, m, M, μ, g)

(4 分) 長方體的長度 L 是多少 (以 v, m, M, μ, g 表示) ?

(d) (3 points) From time $t = 0$ to τ , what is the total work done by the frictional force on the small cube and the rectangular block?

(3 分) 在 $t = 0$ 至 τ 期間，摩擦力對小方塊和長方體所做的功，其總和是多少？



Solution:

(a) The kinetic friction acting between m and M is

$$f = \mu mg$$

The acceleration and velocity of the cube are

$$a = \mu g$$

$$v_m(t) = v - \mu g t \quad [1]$$

On the other hand, the acceleration and velocity of the block are

$$A = \frac{m}{M} \mu g$$

$$V_M(t) = \frac{m}{M} \mu g t. \quad [1]$$

The cube m will stop on the block when $v_m(\tau) = V_M(\tau)$, [1]

$$\Rightarrow v - \mu g \tau = \frac{m}{M} \mu g \tau$$

$$\Rightarrow \tau = \frac{v}{\left(1 + \frac{m}{M}\right) \mu g} = \frac{Mv}{(M + m) \mu g} \quad [1]$$

(b) Assuming the initial position of m is at the origin ($x(0) = 0$)

$$x_m(t) = vt - \frac{1}{2}at^2$$

The distance travelled by the cube is

$$x_m(\tau) = v\tau - \frac{1}{2}a\tau^2 = \frac{v^2}{\left(1 + \frac{m}{M}\right)\mu g} - \frac{v^2}{2\left(1 + \frac{m}{M}\right)^2\mu g} = \frac{(M + 2m)Mv^2}{2(M + m)^2\mu g} \quad [1]$$

And the distance travelled by the block is

$$X = \frac{1}{2}A\tau^2 = \frac{1}{2}\frac{m}{M}\frac{v^2}{\left(1 + \frac{m}{M}\right)^2\mu g} = \frac{mMv^2}{2(M + m)^2\mu g}$$

(c) When the motion stops, the coordinate of the midpoint of the rectangular block is

$$X_{mid} = \frac{L}{2} + \frac{1}{2}A\tau^2 = \frac{L}{2} + \frac{1}{2}\frac{m}{M}\frac{v^2}{\left(1 + \frac{m}{M}\right)^2\mu g} \quad [1]$$

As the cube stops at the midpoint of the rectangular block, we have

$$X_{mid} = \frac{L}{2} + \frac{1}{2}A\tau^2 = \frac{L}{2} + \frac{1}{2}\frac{m}{M}\frac{v^2}{\left(1 + \frac{m}{M}\right)^2\mu g} = x_m(\tau) = \frac{v^2}{\left(1 + \frac{m}{M}\right)\mu g} - \frac{v^2}{2\left(1 + \frac{m}{M}\right)^2\mu g} \quad [1]$$

$$\Rightarrow \frac{L}{2} = \frac{v^2}{\left(1 + \frac{m}{M}\right)\mu g} - \frac{v^2}{2\left(1 + \frac{m}{M}\right)^2\mu g} - \frac{1}{2}\frac{m}{M}\frac{v^2}{\left(1 + \frac{m}{M}\right)^2\mu g}$$

$$\Rightarrow L = 2\frac{v^2}{\mu g}\frac{1}{\left(1 + \frac{m}{M}\right)}\left(1 - \frac{1}{2}\frac{1}{\left(1 + \frac{m}{M}\right)} - \frac{1}{2}\frac{m/M}{\left(1 + \frac{m}{M}\right)}\right) = \frac{v^2}{\mu g}\frac{1}{\left(1 + \frac{m}{M}\right)} = \frac{Mv^2}{(M + m)\mu g} \quad [1]$$

(d) **(Method 1)** The total work done by the frictional force is

$$W = \mu mg\left(-\frac{v^2}{\left(1 + \frac{m}{M}\right)\mu g} + \frac{v^2}{2\left(1 + \frac{m}{M}\right)^2\mu g} + \frac{1}{2}\frac{m}{M}\frac{v^2}{\left(1 + \frac{m}{M}\right)^2\mu g}\right) \quad [1]$$

$$\Rightarrow W = mv^2\left(-\frac{1}{1 + \frac{m}{M}} + \frac{1}{2\left(1 + \frac{m}{M}\right)^2} + \frac{1}{2}\frac{\frac{m}{M}}{\left(1 + \frac{m}{M}\right)^2}\right) \quad [1]$$

$$\Rightarrow W = -\frac{1}{2}\frac{mv^2}{\left(1 + \frac{m}{M}\right)} \quad [1]$$

(Method 2) The total work done by the frictional force is

$$W = \mu mg(-x_m) + \mu mgX = -\frac{\mu mgL}{2} = -\frac{\mu mg}{2} \frac{Mv^2}{(M+m)\mu g} = -\frac{m}{2} \frac{v^2}{\left(1 + \frac{m}{M}\right)} \quad [1 + 1 + 1]$$

(Method 3) By the work-energy theorem, we have

$$W = KE_f - KE_i = \frac{1}{2}(m+M)v_f^2 - \frac{1}{2}mv^2 = \frac{1}{2}(m+M) \frac{m^2v^2}{(m+M)^2} - \frac{1}{2}mv^2 \quad [1]$$

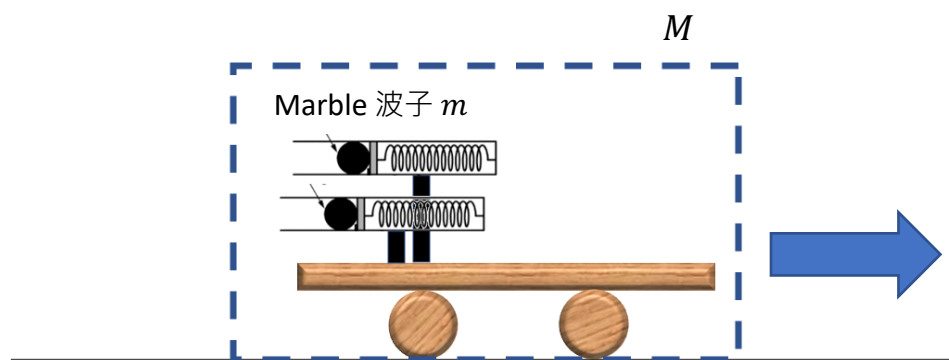
$$\Rightarrow W = \frac{1}{2} \frac{m^2 - m^2 - mM}{m+M} v^2 = -\frac{1}{2} \frac{mv^2}{\left(1 + \frac{m}{M}\right)} \quad [1]$$

Where the final velocity of the block and the cube are

$$v_f = V_M(\tau) = \frac{m}{M} \mu g \tau = \frac{m}{M} \mu g \frac{Mv}{(M+m)\mu g} = \frac{mv}{(M+m)} \quad [1]$$

2. (15 points) A wooden cart is driven by shooting marbles from identical compressed spring launchers mounted on the cart. The cart, including the marbles, has a mass of M and each marble has a mass of m . For simplicity, we assume that the resistance experienced by the cart can be effectively represented by a kinetic friction coefficient μ between the ground and the cart and all other resistance such as air drag can be neglected. The initial velocity of the cart is u before launching any marbles. When a marble is fired by one of the spring launchers, we assume the impact time is very short and is negligible. We also assume all the stored elastic energy goes to the kinetic energy of different parts without generating heat. We are interested on how far the cart can go by firing these marbles.

(15 分) 一架木製車子通過安裝在車身上相同的壓縮彈簧發射器發射波子來推動。包括波子在內的車子的質量為 M ，每粒波子的質量為 m 。為簡單起見，我們假設作用在車子的阻力可以地面與車子間的動摩擦係數 μ 代表，其他阻力例如空氣阻力則可以忽略。波子發射之前，車子的初始速度為 u 。當一個彈簧發射器發射波子時，衝擊時間非常短並且可以忽略不計。我們還假設所有存儲的彈性能都轉化為不同零件的動能且不會生熱。我們感興趣的問題，是通過發射這些波子可以使推車走多遠。



- (a) (7 points) Suppose each spring is compressed by an amount of Δx in length with spring constant k . Find the velocity v for the cart after one marble is launched. What is the distance the cart can elapse before it stops if the initial velocity of the cart is $u = 0 \text{ ms}^{-1}$?

(7 分) 假設每個彈簧的彈簧常數為 k ，並在長度上壓縮 Δx 。求車子在發射一粒波子後的速度 v 。如果車子的初始速度為 $u = 0 \text{ ms}^{-1}$ ，問車子在停止之前行經的距離是多少？

- (b) (8 points) You can fire a second marble at any time you choose, what will be the maximum and minimum distance the cart can elapse in firing two marbles? The initial velocity of the cart is still zero before firing the two marbles. Please take the limit $m \ll M$ for simplification.

(8 分) 你可以在任何時候發射第二粒波子，車子發射兩粒波子後可以行經的最大和最小距離是多少？在發射兩粒波子前，車子的初始速度仍為零。為簡化起見，請假設 $m \ll M$ 。

Solution:

(a) In the frame moving at velocity u , the marble moves at velocity v'_m while the cart moves at velocity v' . The conservation of momentum gives

$$v'_m = -\frac{(M-m)v'}{m}$$

We assume the elastic energy completely goes into the kinetic energy of all parts, i.e.

$$\begin{aligned} \frac{1}{2}k(\Delta x)^2 &= \frac{1}{2}(M-m)v'^2 + \frac{1}{2}mv_m'^2 = \frac{1}{2}(M-m)v'^2 + \frac{1}{2m}(M-m)^2v'^2 \\ \Rightarrow v &= u + v' = u + \frac{\sqrt{km}}{\sqrt{M}\sqrt{M-m}}\Delta x \end{aligned}$$

The deceleration of the cart is μg and the distance elapsed L is

$$L = \frac{v^2 - 0^2}{2\mu g} = \frac{km(\Delta x)^2}{2\mu g M(M - m)}$$

b) After the first marble has been fired, the second marble can be fired when the velocity of the cart decreases to u_2 . At that time, the cart has already travelled by a distance L_1 :

$$L_1 = \frac{v^2 - u_2^2}{2\mu g} \cong \frac{km(\Delta x)^2}{2\mu g M^2} - \frac{u_2^2}{2\mu g}$$

The cart then changes its velocity to $v \cong u_2 + \frac{\sqrt{km}}{M} \Delta x$ by firing the second marble and travels by a distance L_2 :

$$L_2 \cong \frac{1}{2\mu g} \left(\left(u_2 + \frac{\sqrt{km}}{M} \Delta x \right)^2 - 0^2 \right) \cong \frac{1}{2\mu g} \left(u_2 + \frac{\sqrt{km}}{M} \Delta x \right)^2$$

The total distance it travels by firing two marbles is therefore

$$L = L_1 + L_2 = \frac{km(\Delta x)^2}{\mu g M^2} + \frac{\sqrt{km}}{\mu g M} u_2 \Delta x$$

Therefore, the minimum distance is at the case that the second marble is fired after the cart comes at rest:

$$L = \frac{km(\Delta x)^2}{\mu g M^2}$$

If the second marble is fired immediately after the first marble is fired, the total distance is

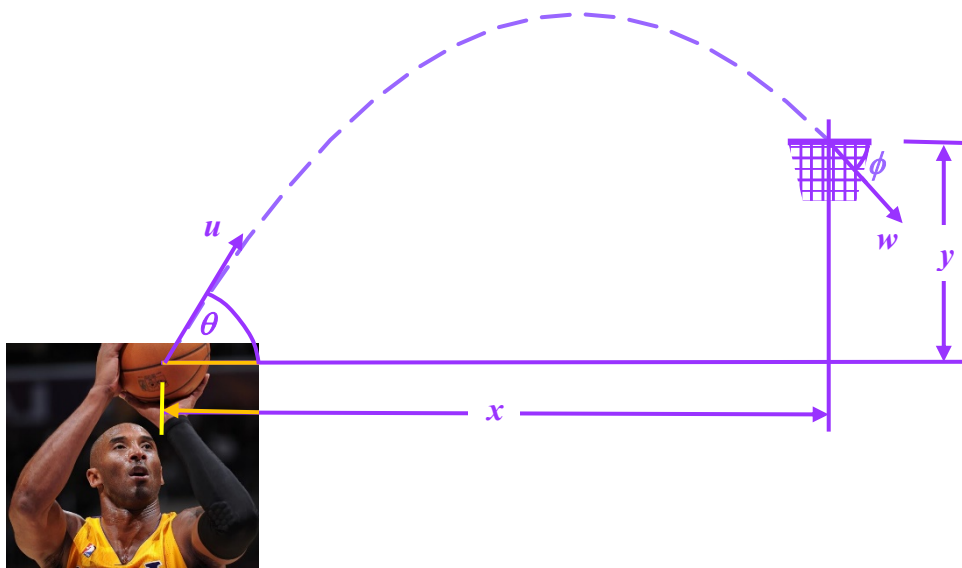
$$L \cong \frac{km(\Delta x)^2}{\mu g M^2} + \frac{\sqrt{km}}{\mu g M} \left(\frac{\sqrt{km}}{M} \Delta x \right) \Delta x = \frac{2km(\Delta x)^2}{\mu g M^2}$$

which is the maximum distance. In fact, if the two marbles are fired at the same time, we can obtain the distance by changing $k \rightarrow 2k, m \rightarrow 2m$ in the answer of part (a) and take $m \ll M$ as

$$L \cong \frac{(2k)(2m)(\Delta x)^2}{2\mu g M^2} = \frac{2km(\Delta x)^2}{\mu g M^2}.$$

Note: this answer goes back to the answer of L for firing the two marbles consecutively only when $m \ll M$ for the reason that we have neglected the dynamics within the duration of impact for simplicity.

3. (20 points) This problem is a tribute to Kobe Bryant, one of the greatest basketball players of all time but whose life was taken in a helicopter accident earlier this year. (20 分) 這題是對高比拜仁(Kobe Bryant)的致敬，他是有史以來最偉大的籃球運動員之一，但他在今年初的一次直升機事故中喪生。



Consider a free-throw shot in a basketball game. He projects the basketball with an initial velocity u . Assume that air resistance is negligible in this problem.

考慮一個籃球比賽中的罰球。他以初始速度 u 投射籃球。假設在此問題中空氣阻力可以忽略不計。

- (a) (4 points) Derive an explicit expression of the angle of inclination θ such that the basketball can hit the center of the basket rim, which is at a horizontal distance x and a vertical distance y .

(4 分) 求傾角 θ 的明確表達式，以使籃球可以擊中籃筐的中心，該中心點在水平距離 x 和垂直距離 y 處。

The projectile motion is given by

$$\begin{aligned} x &= u \cos \theta t, \\ y &= u \sin \theta t - \frac{1}{2} g t^2. \end{aligned}$$

Eliminating t ,

$$y = x \tan \theta - \frac{g x^2}{2 u^2} \sec^2 \theta.$$

Since $\sec^2 \theta = 1 + \tan^2 \theta$, we arrive at a quadratic equation for $\tan \theta$,

$$\begin{aligned} \frac{g x^2}{2 u^2} \tan^2 \theta - x \tan \theta + \frac{g x^2}{2 u^2} + y &= 0. \\ \tan \theta &= \frac{x \pm \sqrt{x^2 - \frac{2 g x^2}{u^2} \left(\frac{g x^2}{2 u^2} + y \right)}}{\frac{g x^2}{u^2}} = \frac{u^2}{g x} \pm \sqrt{\frac{u^4}{g^2 x^2} - \frac{2 y u^2}{g x^2} - 1}. \end{aligned}$$

- (b) (4 points) A physics professor suggested that the best shot can be achieved by shooting the ball at the minimum possible speed, because it has a higher chance of passing through the hoop in case the shot is not accurate and bounces off the rim of the hoop. Derive an expression for the minimum shooting speed and the corresponding shooting angle θ .

(4 分) 一位物理學教授建議，以盡可能低的速率射球為最佳，因為如果射球不準確並且從籃筐彈出，則穿過球籃的機會較大。求最小射球速率及對應發射角 θ 的表達式。

(The physics professor was Peter Brancazio, who passed away this year due to COVID 19 complications.) In general, for a shooting speed u , there are two possible shooting angles. However, when the shooting speed is minimum, there is only one possible shooting angle. Hence

$$\frac{u^4}{g^2 x^2} - \frac{2yu^2}{gx^2} - 1 = 0.$$

This is a quadratic equation for u^2 . The solution is

$$u^2 = \frac{g^2 x^2}{2} \left(\frac{2y}{gx^2} \pm \sqrt{\frac{4y^2}{g^2 x^4} + \frac{4}{g^2 x^2}} \right).$$

$$u^2 = g \left(y + \sqrt{y^2 + x^2} \right).$$

$$u = \sqrt{g \left(y + \sqrt{y^2 + x^2} \right)}.$$

$$\theta = \arctan \frac{u^2}{gx} = \arctan \frac{y + \sqrt{y^2 + x^2}}{x}.$$

- (c) (4 points) For Kobe Bryant, the ball is launched from a height of 2.47 m and the rim of the hoop is 3.05 m above the ground. The horizontal distance x is 4.44 m. Calculate the shooting speed and shooting angle in part (b).

(4 分) 在高比拜仁的情況中，球從 2.47 m 的高度射出。籃筐的邊緣在離地面 3.05 m 的位置，水平距離 x 為 4.44 m。計算 (b) 部分中的射球速率和發射角度。

$$y = 3.05 - 2.47 = 0.58 \text{ m}.$$

$$u = \sqrt{9.8 \left(0.58 + \sqrt{0.58^2 + 4.44^2} \right)} = 7.04 \text{ ms}^{-1}.$$

$$\theta = \arctan \frac{0.58 + \sqrt{0.58^2 + 4.44^2}}{4.44} = 49^\circ.$$

Remark about the dimensions in the question: It has been observed that free throws are shot from a few inches in front of the free throw line, which is 15 feet from the backboard. We thus take x to be 4.44 m.

It has also been observed that shooters release the ball, on average, from a height of approximately 1.25 times the shooter's own height. Since the height of Kobe Bryant is 1.98 m, we estimate that the ball is launched from a height of 2.47 m.

(d) (4 points) Calculate the angle of approach ϕ when the basketball arrives at the hoop for the shot in part (c).

(4 分) 計算 (c) 部分中當籃球到達籃筐時的接近角 ϕ 。

$$v_x = u \cos \theta,$$

$$v_y = u \sin \theta - gt = u \sin \theta - \frac{gx}{u \cos \theta}.$$

Dividing the two equations,

$$\tan \phi = -\frac{v_y}{v_x} = -\tan \theta + \frac{gx}{u^2 \cos^2 \theta} = \frac{gx}{u^2} \tan^2 \theta - \tan \theta + \frac{gx}{u^2}.$$

At the minimum shooting speed,

$$\tan \phi = \frac{gx}{u^2} \left(\frac{u^2}{gx} \right)^2 - \frac{u^2}{gx} + \frac{gx}{u^2} = \frac{gx}{u^2}.$$

$$\phi = \arctan \frac{gx}{u^2} = \arctan \frac{x}{y + \sqrt{y^2 + x^2}} = 41^\circ.$$

(4 points) Another physics professor, being a basketball player himself, suggested that Kobe's shot can be improved by shooting with at a higher angle, so that the angle of approach ϕ is higher and results in a bigger target of the hoop. Calculate the shooting angle θ and shooting speed corresponding to $\phi = 45^\circ$.

(4 分) 另一位物理學教授則建議 (他本人也是一位籃球運動員)，通過以更大的發射角度射球可以改善高比的投籃，因為接近角 ϕ 會增大，從而導致更大的籃筐目標。計算對應於 $\phi = 45^\circ$ 的發射角 θ 和射球速率。

(The physics professor was John Fontanella, author of “The Physics of Basketball”.) From (d),

$$\tan \phi = \frac{gx}{u^2} \sec^2 \theta - \tan \theta.$$

From (a),

$$\frac{gx}{u^2} \sec^2 \theta = 2 \tan \theta - \frac{2y}{x}.$$

Hence

$$\tan \phi = \tan \theta - \frac{2y}{x}.$$

$$\tan \theta = \tan \phi + \frac{2y}{x} = \tan 45^\circ + \frac{2(0.58)}{4.44} = 1.2613.$$

$$\theta = \arctan 1.2809 = 52^\circ$$

$$u = \sqrt{\frac{gx(1 + \tan^2 \theta)}{2 \tan \theta - 2y/x}} = \sqrt{\frac{(9.8)(4.44)(1 + 1.2613^2)}{2(1.2613) - 2(0.58)/(4.44)}} = 7.06 \text{ ms}^{-1}.$$

~ END of PAPER 全卷完 ~