Hong Kong Physics Olympiad 2022 – Paper 1
2022 年香港物理奧林匹克競賽 – 卷一

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18 September, 2022

2022 年 9 月 18 日

Rules and Regulations 競賽規則

1. All questions are in bilingual versions. You can answer in Chinese or English, but only ONE language should be used throughout the paper.
所有題目均為中英對照。你可選擇以中文或英文作答，惟全卷必須以單一語言作答。

2. Please write your 3-digit Contestant Number and English Name on the first page of the answer books.
在答題簿的第一頁上，請填上你的 3 位數字參賽者號碼及英文姓名。

3. The open-ended problems are quite long. Please read the whole problem first before attempting to solve them. If there are parts that you cannot solve, you are allowed to treat the answer as a known answer to solve the other parts.

開放式問答題較長，請將整題閱讀完後再著手解題。若某些部分不會做，也可把它們的答案當作已知來做其他部分。

The following symbols and constants are used throughout the examination paper unless otherwise specified:

除非特別注明，否則本卷將使用下列符號和常數：

Gravitational acceleration on Earth's surface 地球表面重力加速度	g	9.8 m s^{-2}
Gravitational constant 重力常數	G	6.67×10^{-11} m^{3}kg^{-1}s^{-2}

The following identities may be useful in the competition:

1.

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x, \qquad \text{when } |x| \ll 1$$

2.

$$\sin x \approx \tan x \approx x. \qquad \text{when } x \ll 1 \text{ is measured in radian}$$

Multiple Choice Questions

Multiple Choice Questions (2 Marks for each correct answer. 0 Marks for each wrong answer.)

選擇題(每個正確答案得 2 分。 每答錯一題得 0 分)

1. As shown below, Bob is using the rope through a fixed pulley to move a box with constant speed v. The coefficient of kinetic friction between the box and the ground is $\mu < 1$. Assume that the fixed pulley is massless and there is no friction between the rope and the fixed pulley. While the box is moving, which of the following statements is correct?

1. 如下圖所示， Bob 正在使用繩索通過一個固定滑輪以恆定速率 v 移動一個盒子。盒子與地面之間的動摩擦系數為 $\mu < 1$。 假設固定滑輪是無質量的，繩索和固定滑輪之間沒有摩擦。 當盒子在移動時，下列哪個說法是正確的？
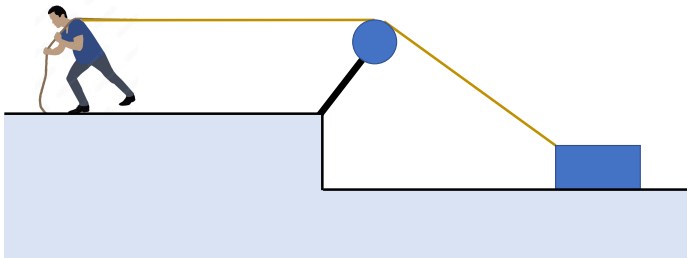
A. The magnitude of the force on the rope is constant. 繩索上的力的值不變。

B. The magnitude of friction between the ground and the box is decreasing. 地面和盒子之間的摩擦力正在減小。

C. The magnitude of the normal force of the ground on the box is increasing. 地面對盒子的法向力的值正在增加。

D. The pressure of the box on the ground is increasing. 箱子作用在地面上的壓力越來越大。

E. The pressure of the box on the ground is constant. 箱子作用在地面上的壓力不變。

Solution:

Let the tension on the rope is T. For constant speed, the net force on the box must be zero.

$$T \cos \theta = \mu N$$

$$N + T \sin \theta = mg$$

$$\Rightarrow T \cos \theta = \mu(mg - T \sin \theta)$$

$$\Rightarrow T = \frac{\mu mg}{\cos \theta + \mu \sin \theta}$$

$$N = mg - T \sin \theta = mg \left(1 - \frac{\mu}{\mu + \cot \theta}\right)$$
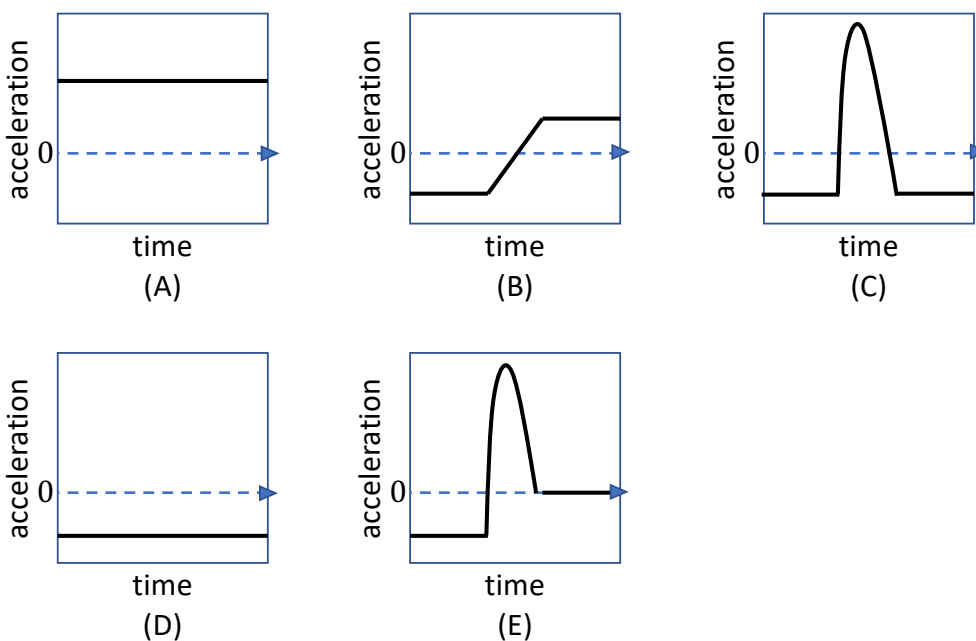
2. An inelastic ball is dropped from a height and sticks on the ground. Which graph best represents the acceleration of the center of mass of the ball as a function of time?

2. 一個無彈性的球從高處落下並粘在地上。 哪張圖最能代表球的質心加速度隨時間的變化？

3. A basketball is dropped from the height and bounces on the ground. Considering only the ball just before and just after the bounce, which of the following statements must be true?

3. 一個籃球從高處掉下，並從地面反彈。 只考慮在剛反彈之前和之後的球，以下哪個陳述必為正確？

A. The momentum and the total mechanical energy of the ball are conserved. 球的動量和總機械能是守恆的。

B. The momentum of the ball is conserved, but not the kinetic energy. 球的動量守恆，但動能不守恆。

C. The total mechanical energy of the ball is conserved, but not the momentum. 球的總機械能守恆，但動量不守恆。

D. The kinetic energy of the ball is conserved, but not the momentum. 球的動能守恆，但動量不守恆。

E. The momentum of the ball is not conserved. 球的動量不守恆。

4. A ball enters a 90° bend tube lying on the ground (as shown in the figure). Both the ball and the bend tube can slide freely on the ground. What is the direction of the velocity of the ball (relative to the ground) when it exits the tube?

4. 如圖所示，一個球進入一條放在地面上的 90° 彎管。 球和彎管都可以在地面上自由滑動。 當球離開管子時，球相對於地面的速度方向是什麼？

A.

B.

C.

D.

E.
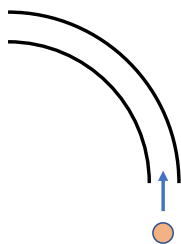
5. A projectile is launched at an angle of 25.0° from the horizontal. Is there any point on the trajectory where $\vec{v}$ and $\vec{a}$ are parallel to each other? If so, where?

5. 炮彈以與水平面成 25.0° 的角度發射。炮彈的軌跡上是否存在一個點，於該點處 $\vec{v}$ 和 $\vec{a}$ 相互平行？ 如果有，這個點在哪裡？

A. An instant before impact on the ground. 撞擊地面的前一刻

B. At the highest point. 在最高點。

C. They are never parallel. 它們從不平行。

D. An instant after launch. 發射後的瞬間。

E. Somewhere in the middle when the projectile is going down. 位於炮彈下降時的某個位置。

6. A cannon car can fire a cannonball in any direction at a fixed speed u with respect to itself. Now the cannon car, initially at rest, can move towards North by firing two cannon balls to propel itself. Ignore all frictions. Which of the following situations will give the cannon car the highest speed after firing the two cannonballs?

6. 一砲車可以向任何方向發射砲彈。 假設砲彈總是以相對於砲車的速率 u 發射。 今該砲車可以透過發射兩枚砲彈以從初始靜止變為最後向北運動。 忽略所有的摩擦。 以下哪種情況會使砲車在發射完兩枚砲彈後獲得最高速率？

A. Fire both cannonballs horizontally towards South at the same time. 同時向南水平方向發射兩枚砲彈。

B. Fire one cannonball horizontally towards South, then wait for 5 seconds and fire another one towards South again. 向南水平方向發射一枚砲彈，等待 5 秒再向南發射另一枚。

C. Fire one cannonball at 45° to the ground towards South, then wait for 5 seconds and fire another one towards South again. 向南方並與以地面成 45° 的方向發射一枚砲彈，等待 5 秒再向南發射另一枚砲彈。

D. Fire one cannonball horizontally towards South-West and another towards South-East at the same time. 同時向西南和東南水平方向各發射一枚砲彈。

E. Both A and B. will give the car the same speed. A 和 B 都會給予砲車相同的速率。

7. Consider the following two cases:

Case A: A boy is pulling himself up on a movable platform hung on a frictionless pulley at a constant speed.

Case B: The boy on the platform is pulled by another boy standing on the ground.

Suppose the total weight of the boy on the platform and the platform is Mg. The rope is massless. The boy and the platform eventually moved 2 m vertically upwards. Which of the following statements is/are true?

I. The pulling force applied by the boy in case A is half of the force applied by the boy on the ground in case B.
II. To move up 2 m, the work done by the boy in case A is less than the work done by the boy on the ground in case B.
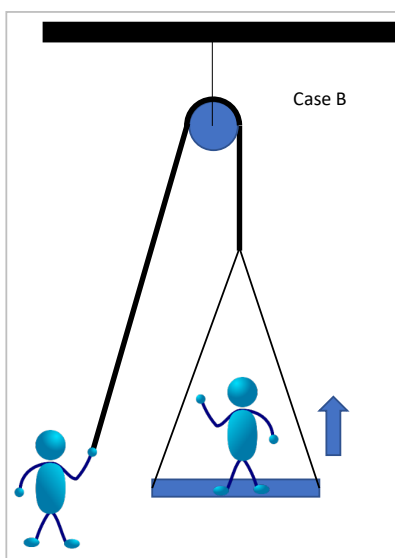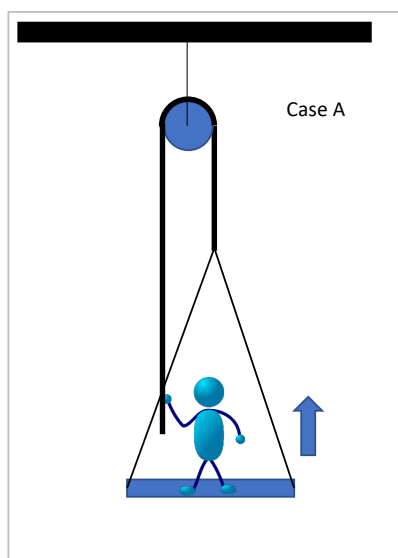III. The tension on the rope in both cases are the same.

7. 考慮以下兩種情況：

情況 A：一個男孩在一個懸掛在無摩擦滑輪上的可移動平台上以匀速拉起自己。

情況 B：平台上的男孩被另一個站在地面上的男孩拉起。

假設平台和平台上男孩的總重量為 Mg。 繩索沒有質量。男孩和平台最終垂直向上移動 2m 。 以下哪項陳述是正確的？

I. 在情況 A 中的男孩施加的拉力是情況 B 中在地面上的男孩施加的拉力的一半。

II. 為使物體向上移動 2 m，情況 A 中的男孩所做的功小於情況 B 中在地面上的男孩所做的功。

III. 兩種情況下繩索上的張力相同。

Case A

Case B

A.   I only   B.   I and II only   C.   II and III only

D.   I, II and III   E.   None of them

Answer: A

Solution: The tension at any point of the rope is the same. In Case A, the rope is pulling the boy and the platform system at two different points. At each point, the tension of the rope pull the boy-platform system up. Therefore, the tension is half of the weight of the system. The pulling force equals to the tension of the rope that is half of the total weight of the system.

For Case B, there is only one point of contact of the system and the rope. The tension of the rope must equal to the total weight of the system. Therefore, the pulling force in Case B is twice as big as that in Case A.

Despite the force are different, the work done in both cases is the same. Since in both cases the work is done to compensate the work done by gravity which only depends on the height the system moves.

Alternatively, we can see that the force in case A is half of that of in case B. But to lift the system up by 1m, the rope needed to be pulled for 2m. As work done equals to force times displacement, the two cases are the same in term of work done.

8. Four identical springs, with spring constant k, are mounted on 3 rigid and vertical thin plates as following in their equilibrium situation. Now, we compress the whole assembly as a whole in the direction perpendicular to the plates. What is the effective spring constant as a whole? Assume the masses of both the springs and the plates can be neglected and the displacement is small.

8. 四個相同的彈簧，彈簧常數為 k，分別安裝在 3 個剛性垂直薄板上，整個系統維持平衡狀態。 現在，我們將整個系統作為一個整體在垂直於薄板的方向上壓縮。 整體上的有效彈簧常數是多少？ 假設彈簧和薄板的質量都可以忽略，而且位移很小。
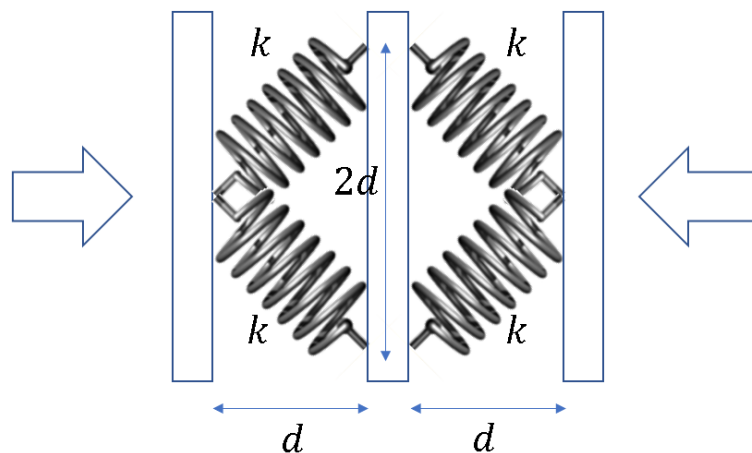
A. $k/2$ B. $k/\sqrt{2}$ C. k

Solution: A

The total stored energy is

$$E = 4 \times \frac{1}{2}k\left(\sqrt{\left(d + \frac{x}{2}\right)^2 + d^2} - \sqrt{2}d\right)^2$$
$$\cong 4 \times \frac{1}{2}k\left(\frac{1}{\sqrt{2}}\frac{x}{2}\right)^2 = \frac{1}{2}\frac{k}{2}x^2$$

Therefore the effective spring constant is $k/2$

9. A lifeguard at point A on the beach, 10 m from the coastline, is going to save a swimmer in the sea at point B, also 10 m from the coastline. The alongshore distance is 2m between points A and B, as shown in the figure (not in scale). The lifeguard can run at 6 m/s on the beach and swim at 2 m/s in the sea. Suppose the lifeguard will go in a straight line from A to C, a point on the coastline and another straight line from C to B. Find the distance x, i.e. the alongshore distance between A and C if he would like to minimize the total time to reach point B. You may use the approximation formula written on P.2.

9. 在距離海岸線 10 m 的海灘上 (A 點)有一救生員,他正準備拯救在海中的一名游泳者(B 點),B 點也距離海岸線 10 m。 A 點和 B 點之間的沿岸距離為 2m,如圖所示(未按比例)。 救生員可以在沙灘上以 6 m/s 的速度奔跑,在海中以 2 m/s 的速度游泳。 設 C 點為海岸線上的一個點,救生員將從 A 點以直線走到 C 點 ,再從 C 點以直線游到 B 點。如果他想以最短的時間到達 B 點,求距離 x,即 A 和 C 之間的沿岸距離。你可以使用寫在 P.2 上的近似值公式。
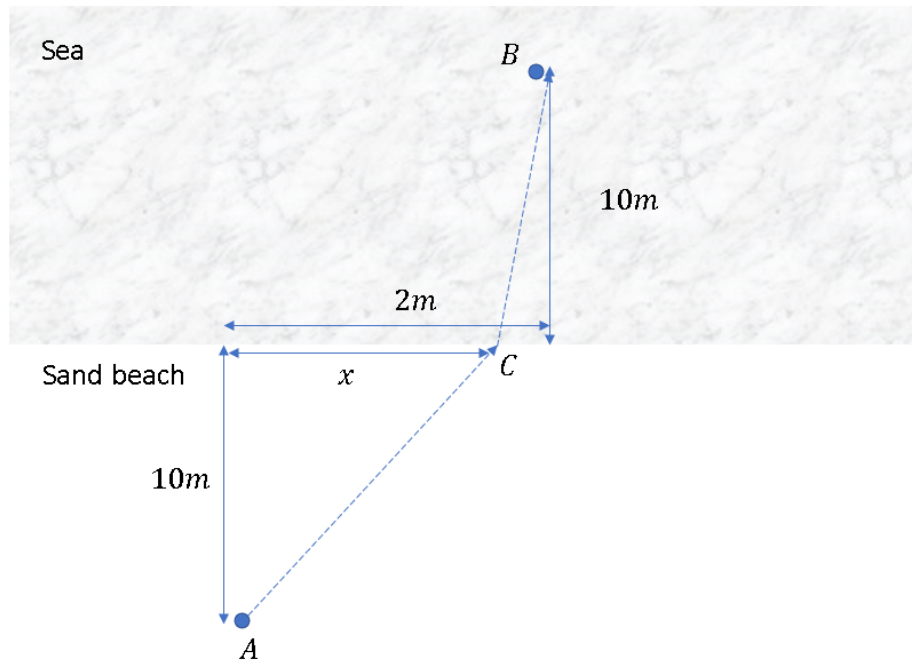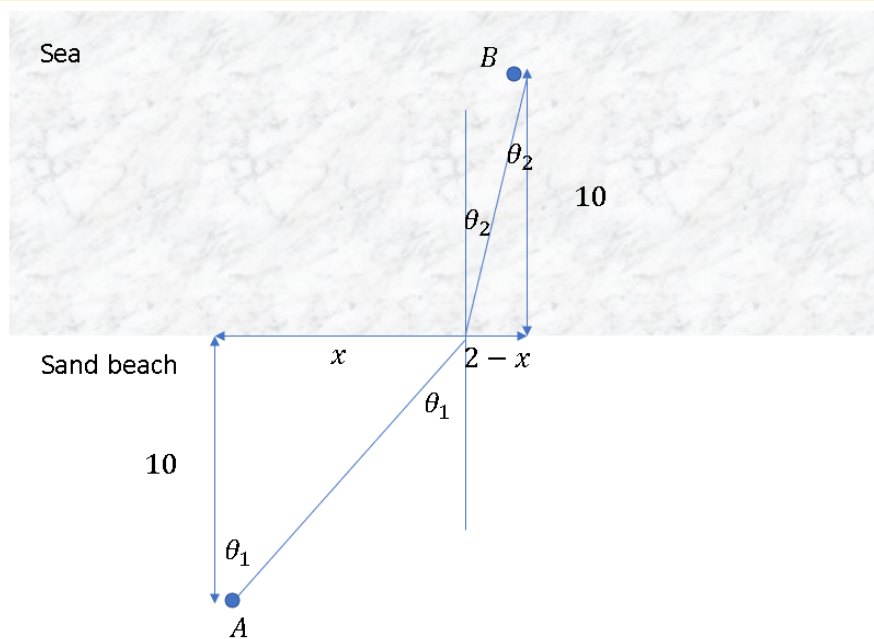
A. 0 m   B. 0.5 m   C. 1.0 m   D. 1.5 m   E. 2.0 m

Solution:

Method 1:

We define angle θ_1 and θ_2 below, in analogy to Snell's law.

$$\frac{\sin \theta_2}{\sin \theta_1} = \frac{v_2}{v_1} = \frac{2}{6}$$

As the horizontal distance is much smaller than the vertical distances, the angles θ_1 and θ_2 are very small, i.e.

$$\frac{2 - x}{x} \cong \frac{1}{3}$$

giving

$$x = 1.5 \text{ m}$$

Method 2:

Algebraic approach:

The total travel time is

$$
\begin{aligned}
T &= \frac{\sqrt{100 + x^2}}{6} + \frac{\sqrt{100 + (2 - x)^2}}{2} \\
&\cong \frac{10 + \frac{x^2}{20}}{6} + \frac{10 + \frac{(2 - x)^2}{20}}{2} \\
&= \frac{1}{6}\left(40 + \frac{x^2 + 3(2 - x)^2}{20}\right)
\end{aligned}
$$

Method 2a: quadratic minimum

$$T = \frac{1}{120}(4x^2 - 12x + 812) = \frac{1}{30}(x^2 - 3x + 203) = \frac{1}{30}\left((x - 1.5)^2 + 203 - 1.5^2\right)$$

Hence, T is minimum when $x = 1.5 m$.

Method 2b: using derivative

$$\frac{dT}{dx} = \frac{1}{120}\big(2x - 6(2 - x)\big) = 0$$
$$\Rightarrow x = 1.5 \text{ m}$$

10. The escape velocity is the minimum speed needed for an object to escape from the gravitational influence of a planet without burning fuel. The acceleration due to gravity on the surface of an unknown planet X is given as 7.44 m/s^2 and the escape velocity on the surface is given as 10.06 km/s. Estimate the radius of the Planet X.

10. 逃逸速度是物體在不燃燒燃料的情況下逃離行星引力影響所需的最小速度。 在一個未知的行星 X 上，

地面重力加速度為 7.44 m/s^2，地表逃逸速度為 10.06 km/s。 估算行星 X 的半徑。

A. 1.7×10^6 m   B. 3.4×10^6 m   C. 5.0×10^6 m
D. 6.8×10^6 m   E. 13.6×10^6 m

Solution: D

11. A 15 kg uniform ladder, with a length 2.5 m, is at rest at the corner between a vertical wall and the ground at an angle of $\theta = 60°$ Suppose the vertical wall is smooth. Find the frictional force on the ground to keep the ladder at rest without sliding.

11. 一個 15 kg 的均勻梯子，長 2.5 m，以 $\theta = 60°$ 角靜止在垂直牆與地面之間的拐角處。 假設垂直牆是光滑的。 找到地面上的摩擦力，使梯子保持靜止而不滑動。
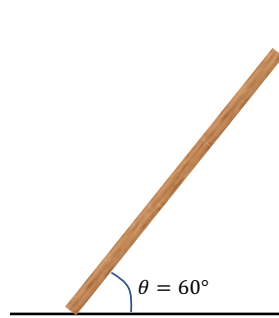
A. 36.8 N   B. 42.4 N   C. 84.9 N   D. 127.3 N   E. None of the above
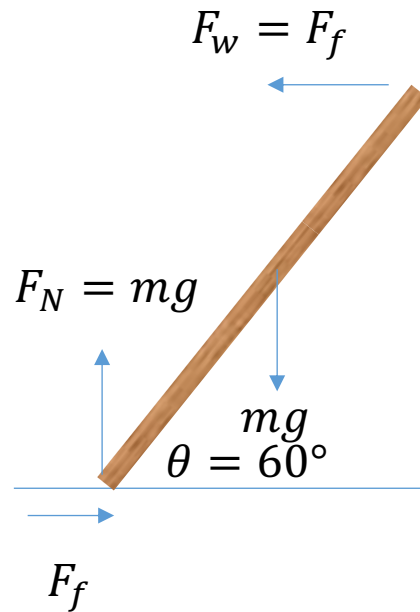
The force diagram is drawn above. To balance force in the horizontal direction, the normal reaction force from the wall F_W equal to the frictional force F_f. The balance of moment about the point of contact on the ground can be expressed as

$$\frac{mgL}{2}\cos\theta = F_w L \sin\theta$$

where L is the length of the ladder. Therefore, we have

$$F_f = \frac{mg}{2}\cot\theta = \frac{15}{2}9.8\cot\frac{\pi}{3} = 42.4 \text{ N}$$

12. Although a photon does not have a mass, it carries a momentum p along its propagation direction, which is related to its energy E by $p = E/c$ where c is the light speed. A solar sail is a proposed method of spacecraft propulsion by shining sunlight on it to exert pressure to push the sail. Suppose the area of the sail is A and there are n photons per unit area per second for the sunlight. What is the maximum resultant force on the spacecraft by the sunlight? Hint: assume the sail is a perfect mirror to reflect the light off completely, i.e. 100% reflection of light.

12. 雖然光子沒有質量，但它沿其傳播方向攜帶動量 p，並與它的能量 E 有關：$p = E/c$ 其中 c 是光速。 太陽帆是一種航天器的推進方法，通過將陽光照射在其上施加壓力來推動帆。 假設帆的面積為 A，而且太陽光中每單位面積每秒有 n 個光子。 請問太陽光對航天器的最大合力是多少？ 提示：假設帆是一面完美的鏡子，可以完全反射光線 (即 100% 光線反射)。

A. $E/\sqrt{A}$ B. $2E/\sqrt{A}$ C. pnA
D. $2pnA$ E.   None of the above (以上皆不是)

13. A tuning fork undergoes simple harmonic motion at one of its ends and fires a middle C musical note at a fixed frequency of 261 Hz. The simple harmonic motion has a maximum amplitude of 0.4 mm. Find the acceleration when the end has a displacement of +0.2 mm.

13. 音叉在其一端進行簡諧運動,並以 261 Hz 的固定頻率發出中央 C 音符。 簡諧運動的最大振幅為 0.4 mm。 求當音叉末端的位移為 +0.2 mm 時的加速度。

A. $+537.9 \text{ m/s}^2$ B. -537.9 m/s^2 C. $+268.9 \text{ m/s}^2$
D. -268.9 m/s^2 E. 0 m/s^2

14. You are given a large collection of identical heavy balls and lightweight rods. When two balls are placed at the ends of one rod and interact through their mutual gravitational attraction (as is shown on the left), the compressive force in the rod is F. Next, four balls and four rods are placed at the vertexes and edges of a square (as is shown on the right). What is the compressive force in each rod in the latter case?

14. 你有大量相同的球和輕質桿。當兩個球放在一根桿的兩端並通過它們的相互引力相互作用(如左圖所示)時,桿中的壓縮力為 F。 接下來,在正方形的頂點和邊緣放置四個球和四個桿(如右圖所示)。 在這一種情況下,每根桿的壓縮力是多少?
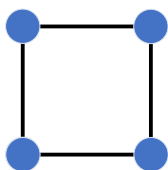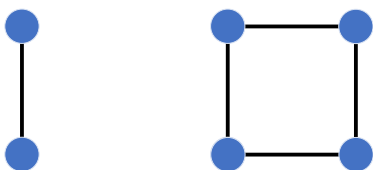
A. F B. $\sqrt{2}\,F$ C. F D. $\left(1 + \frac{1}{2\sqrt{2}}\right)F$ E. $2\sqrt{2}F$

Solution:

The magnitude of the resultant force on ball 1 due to the gravity of the other 3 balls is (pointing along the diagonal direction):

$$F_{grav} = 2 \times F \times \cos\frac{\pi}{4} + \frac{F}{2} = \frac{F}{2}\left(1 + 2\sqrt{2}\right)$$

To get the compressive force in a rod, we know that the net force on ball 1 vanished. We have

$$2F_{rod}\cos\frac{\pi}{4} = F_{grav}$$

$$\Rightarrow F_{rod} = \frac{F_{grav}}{\sqrt{2}} = \frac{F}{2\sqrt{2}}\left(1 + 2\sqrt{2}\right) = F\left(1 + \frac{1}{2\sqrt{2}}\right)$$

Answer D.

15. A block of mass $m = 3.0$ kg slides down one ramp and then up a second ramp smoothly without flipping. The coefficient of kinetic friction between the block and each ramp is $\mu_k = 0.40$. The block begins at a height $h_1 = 1.0$ m above the horizontal. Both ramps are at a 30° incline above the horizontal. To what height h_2 above the horizontal does the block rise on the second ramp?

15. 一塊質量為 $m = 3.0$ kg 的木塊滑下一個坡道,然後平穩地滑上第二個坡道而不會翻轉。 木塊與每個斜坡之間的動摩擦系數為 $\mu_k = 0.40$。木塊初始位於水平面以上 $h_1 = 1.0$ m 的高度。 兩個坡道均與水平面成 30°。問木塊在第二個坡道上可升到水平面上的高度 h_2 是多少?
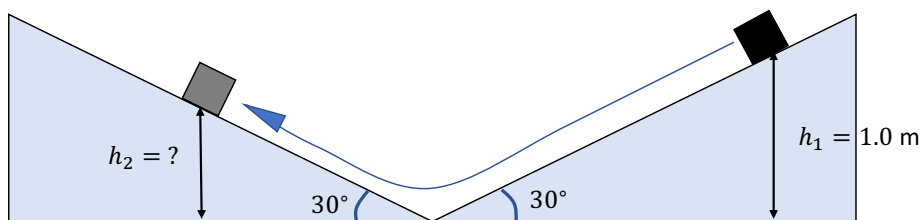
A. 0.18 m   B. 0.52 m   C. 0.59 m   D. 0.69 m   E. 0.71 m

Solution:

The kinetic friction acting on the block during the sliding is

$$F_k = \mu_k N = \mu_k mg \cos\theta$$

By the energy-work theorem

16. Two cannons are arranged vertically under the Earth's gravity, with the lower cannon pointing upward (towards the upper cannon) and the upper cannon pointing downward (towards the lower cannon), 200m above the lower cannon. They fire a projectile at the same time. The initial speed of the projectiles are both equal to 50m/s. How long after the cannons fire do the projectiles collide?

16. 兩門大砲在地球的引力作用下垂直排列，下方大砲向上（朝向上方大砲）。上方大砲向下（朝向下方大砲），並位於在下方大砲上方200 m 處。 它們同時發射砲彈。 砲彈的初始速度均為50 m/s。砲彈在開火後多久會發生碰撞？

200 m
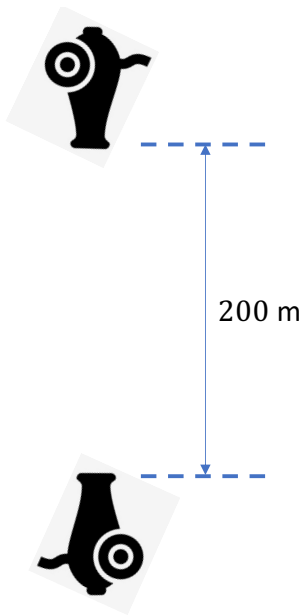
A. 0.5 s   B. 1.0 s   C. 1.5 s   D. 2.0 s   E. 4.0 s

17. Following the above question, how far above the lower cannon do the projectiles collide? Choose the closest value.

17. 承接上題，砲彈在下方大砲上方多遠相撞？ 選擇最接近的值。

A. 50 m   B. 60 m   C. 80 m   D. 100 m   E. 120 m

10. We have

$$x_0 - vt - \frac{1}{2}gt^2 = vt - \frac{1}{2}gt^2$$

$$\Rightarrow t = \frac{x_0}{2v} = 2\ s$$

11. The distance beneath the top cannon is

$$h = x_0 - vt - \frac{1}{2}gt^2 = 200 - 100 - 2g = 80.4\ m$$

18. A 0.100 kg object is released from rest at the height of 3.30 m above the ground and slides along a frictionless track with a loop of diameter 3.00 m. While the object is moving along the loop, it can be considered to be rotating about the center of the loop. When it is at point P (the far-right-most position of the loop), what is the net force exerting on the object (to 3 sig. fig.)?

18. 一個 0.100 kg 的物體在離地 3.30 m 的高度從靜止狀態起動，並沿著一軌道無摩擦滑動。軌道中有一直徑為 3.00 m 的圓環。當物體沿環滑動時，可以認為它是圍繞環中心旋轉。 當它在 P 點（環的最右側位置）時，施加在物體上的淨力是多少（到 3 sig. fig.）？
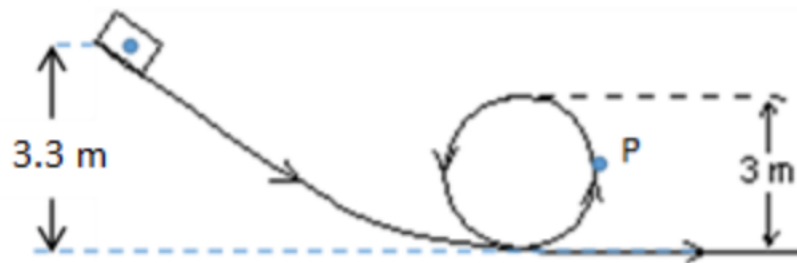
A. 2.35 N   B. 0.98 N   C. 2.55 N   D. 2.04 N   E. 1.56 N

Solution:

The velocity at the point P is

$$\frac{1}{2}mv^2 = mg \times 1.8 \Rightarrow v^2 = 3.6\ g$$

The centripetal force is

$$F_c = \frac{mv^2}{R} = 0.1 * 3.6 * \frac{g}{1.5} = 2.35\ N$$

The total force

$$F_{net} = \sqrt{F_c^2 + (mg)^2} = 2.55\ N$$

19. A car heading north collides at an intersection with a truck heading east. The mass of the truck is twice the mass of the car. If they lock together and travel at 28 m/s at 46° north of east just after the collision, how fast was the car initially travelling? Assume that any forces exerted by objects other than the car or the truck are negligible.

19. 一輛向北行駛的汽車在一個十字路口與一輛向東行駛的卡車相撞。 卡車質量是汽車質量的兩倍。 如果它們在碰撞後鎖定在一起並以 28 m/s 的速度 向東部以北 46°行駛，那麼汽車最初行駛的速度有多快？ 假設任何不是由該汽車或卡車引致的力都可以忽略不計。
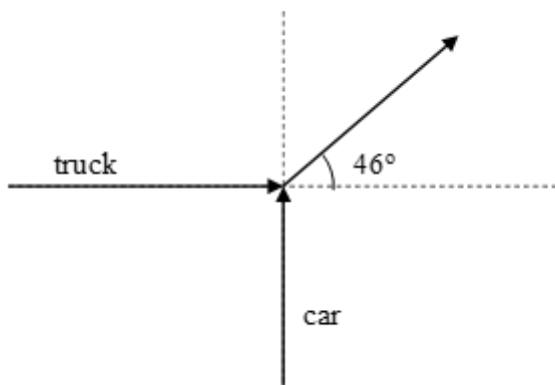
A. 60 m/s

B. 20 m/s

C. 30 m/s

D. 40 m/s

E. 58 m/s

20. A 5.00-kg box is sliding down along an incline. The incline makes an angle of 30.0° to the horizontal and the friction between the box and the inclined surface is non-zero. The box is sliding at a constant speed of 0.60 m/s. Find the magnitude of the power by the friction on the box.

20. 一個 5.00 公斤的箱子沿著斜面滑下。 斜面與水平面成 30.0° 角，盒子與斜面之間的摩擦力不為零。 盒子以 0.60 m/s 的恆定速度滑動。求作用在盒子上的摩擦力的功率大小。

A. 14.7 W   B. 29.4 W   C. 25.5 W   D. 0 W   E. 21.3 W

ANS: 14.7 W

Solution:

The power by the gravitational force is positive and is 100& dissipated by the friction. Therefore, the power by friction is negative to the power by gravity.

$$P_{friction} = -P_{gravity}$$
$$|P_{friction}| = |P_{gravity}| = mgv \sin 30° = 14.7W$$

21. An art sculpture made of material with uniform density is placed on the horizontal ground as shown below. The sculpture is formed by three identical bars with length *L, each with* a mass *M*. The bars are firmly attached together. Neglect the thickness of the bar. Find the difference in the normal force, $\Delta N = N_2 - N_1$, exerting on the sculpture.

21. 一個由密度均勻的材料製成的藝術雕塑被放置在水平地面上，如下圖所示。 雕塑由三根長度為 L 的相同的桿組成，每根桿的質量為 M。這些桿牢固地連接在一起。 可忽略鋼筋的厚度。求作用在雕塑上的法向力的差異，$\Delta N = N_2 - N_1$。
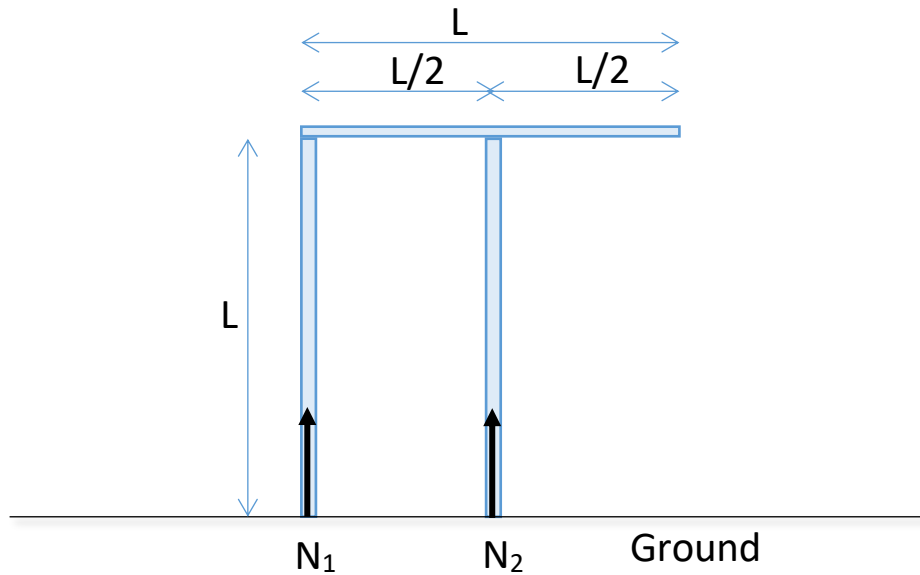
A. $\Delta N = Mg$

B. $\Delta N = 2Mg$

C. $\Delta N = Mg/2$

D. $\Delta N = 3Mg$

E. $\Delta N = 0$

ANS: Mg

Solution:

The center of mass of the system is at (measuring from the left leg)

$$x_{cm} = \frac{m * 0 + m * \frac{L}{2} + m * \frac{L}{2}}{3m} = \frac{L}{3}$$

Conditions for stable equilibrium are that net force and net torque about its center of mass are both zero.
$$F_{net} = 0 \text{ and } \tau_{net} = 0$$

From $F_{net} = 0$, we have $N_1 + N_2 - 3mg = 0$ (1)

From $\tau_{net} = 0$, we have $N_1 * \frac{L}{3} = N_2 * \left(\frac{L}{2} - \frac{L}{3}\right) \Longrightarrow N_2 = 2 N_1$ (2)

Combining (1) and (2), we have $N_2 = 2mg$ and $N_1 = mg$. Thus, the difference is $\Delta N = mg$.

22. A motorcycle moves with constant speed v along a circular track with radius R. The rider leans his body and the motorcycle inwards so that the line joining the track-motorcycle contact point and the center of mass of the whole system (rider + motorcycle) makes an angle θ with the vertically upward direction, as shown in the figure below.
Find θ if $v = 50$ m/s, R = 100 m, and M = 200 kg, where M is the total mass of the motorcycle and the rider.

22. 電單車沿半徑為 R 的圓形軌道以恆速 v 運動。騎手將身體和電單車向內傾斜，使軌道和摩托車的接觸點與整個系統（騎手+摩托車）質心的連線與垂直向上方向形成夾角 θ，如下圖所示。

如果 $v = 50$ m/s、$R = 100$ m 和 $M = 200$ kg，求 θ。 其中 M 是摩托車和騎手的總質量。

A.　　　$\theta = 35$ 　　　　　　B.　　　$\theta = 48$　　　　　　　C.　　　$\theta = 57°$

D.　　　$\theta = 69$　　　　　E.　　　$\theta = 73°$

Solution
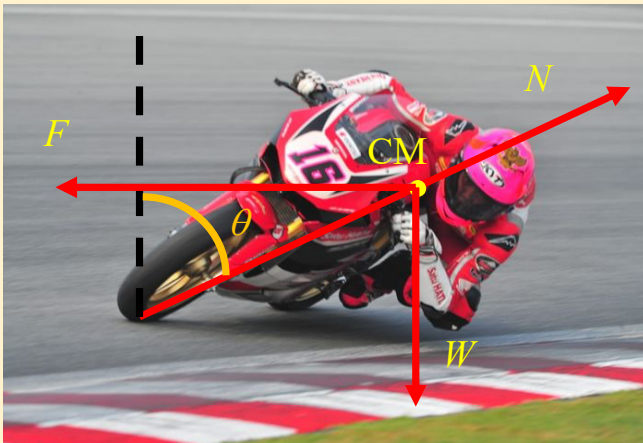
Method 1:

In inertial ground frame,

$$\begin{cases} N \sin \theta = ma = m\dfrac{v^2}{R} \\ N \cos \theta - W = 0 \end{cases}$$

$$\begin{cases} N \sin \theta = m\dfrac{v^2}{R} \\ N \cos \theta = mg \end{cases}$$

23. Two point particles A and B move on the x-y plane. There may be interacting forces between the two particles, but the system as a whole is isolated. At a certain moment, particle A is at $\vec{r}_A = \hat{\imath}$ and particle B is at $\vec{r}_B = \hat{\jmath}$, where $\hat{\imath}$ and $\hat{\jmath}$ are the unit vectors along the x and y direction respectively. Denote the respective accelerations of A and B at this moment by $\vec{a}_A$ and $\vec{a}_B$. Which of the following is/are possible?

23. 兩個粒子 A 和 B 在 x-y 平面上移動。 兩個粒子之間可能存在相互作用力，但整個系統是孤立的。 在某個時刻，粒子 A 位於 $\vec{r}_A = \hat{\imath}$，粒子 B 位於 $\vec{r}_B = \hat{\jmath}$ (其中 $\hat{\imath}$ 和 $\hat{\jmath}$ 分別是沿 x 和 y 方向的單位向量)。 用 $\vec{a}_A$ 和 $\vec{a}_B$ 表示此時 A 和 B 各自的加速度。 以下哪一項是可能的？

I. $\vec{a}_A = -\hat{\imath} + 3\hat{\jmath}, \vec{a}_B = \hat{\imath} + 3\hat{\jmath}$

II. $\vec{a}_A = \hat{\imath} - 3\hat{\jmath}, \vec{a}_B = 3\hat{\imath} - 9\hat{\jmath}$

III. $\vec{a}_A = 3\hat{\imath} - \hat{\jmath}, \vec{a}_B = -6\hat{\imath} + 2\hat{\jmath}$

IV. $\vec{a}_A = 2\hat{\imath} - 2\hat{\jmath}, \vec{a}_B = -3\hat{\imath} + 3\hat{\jmath}$

A.   III only   B.   IV only   C.   III and IV only

D.   II, III, and IV only   E.   I, II, III, and IV

24. A point object is under simple harmonic oscillation on the x-axis centered at the origin. The amplitude is A. At time $t = \tau$, its position is $x = A/2$. At time $t = 2\tau$, its position is $x = -A/2$. What is its position at $t = 3\tau$?

24. 質點在以原點為中心的 x 軸上處於簡諧振動。 幅度為 A。在時間 $t = \tau$，其位置為 $x = A/2$。 在時間 $t = 2\tau$，其位置為 $x = -A/2$。 求它在 $t = 3\tau$ 時的位置。

?

I. $-A$ II. $-A/2$ III. 0   IV. $A/2$

A.   I only   B.   II only   C.   IV only

D.   I or IV   E.   II or III

Solution:

Graphical method:
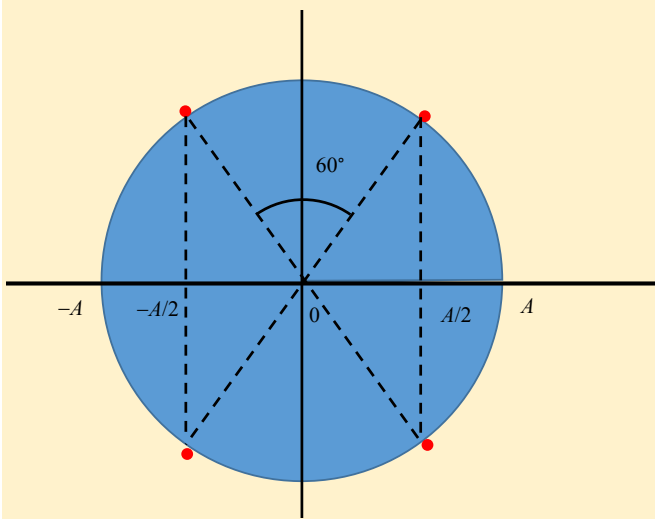
It can be seen that to go from $A/2$ at τ to $-A/2$ at 2τ, the angle traversed in time τ must be $\pm 60° + 360°n$ or $180° + 360°n$.

Starting in the 1st quadrant, with $60° + 360°n$, the position should be $-A$.

Starting in the 1st quadrant, with $180° + 360°n$, the position should be $A/2$.

Starting in the 4th quadrant, with $180° + 360°n$, the position should be $A/2$.

Starting in the 4th quadrant, with $-60° + 360°n$, the position should be $-A$.

Algebraic method:

$$x(t) = A\cos(\omega t + \phi)$$

$$\begin{cases} \cos(\omega\tau + \phi) = 1/2 \\ \cos(2\omega\tau + \phi) = -1/2 \end{cases}$$

$$\begin{cases} \omega\tau + \phi = \pm 60° + 360°m \\ 2\omega\tau + \phi = \pm 120° + 360°n \end{cases}$$

Case 1:

$$\begin{cases} \omega\tau + \phi = 60° + 360°m \\ 2\omega\tau + \phi = 120° + 360°n \end{cases}$$

$$\begin{cases} \omega\tau = 60° + 360°(n - m) \\ \phi = 360°(2m - n) \end{cases}$$

$$\cos(3\omega\tau + \phi) = -1$$

Case 2:

$$\begin{cases} \omega\tau + \phi = 60° + 360°m \\ 2\omega\tau + \phi = -120° + 360°n \end{cases}$$

25. Two tennis balls A and B with the same mass, move with respective velocities u_A and u_B on the x-axis. B is located in the $+x$ direction with respect to A and $u_A > u_B$, so that A will hit B. After the collision, the final velocities of A and B are v_A and v_B, respectively. What are the possible values of v_A? Ignore all external forces acting on A and B.

25. 兩個質量相同的網球 A 和 B 在 x 軸上以速度 u_A 和 u_B 分別運動。 B 處於 A的 $+x$ 方向並且 $u_A > u_B$。因此 A 會撞到 B。碰撞後，A 和 B 的最終速度分別為 v_A 和 v_B。 v_A 的可能值範圍是多少？ 忽略作用在 A 和 B 上的所有外力。

A. $0 \leq v_A \leq u_A$ B. $0 \leq v_A \leq \frac{u_A + u_B}{2}$ C. $u_B \leq v_A \leq \frac{u_A + u_B}{2}$

D. $u_B \leq v_A \leq u_A$ E. $\frac{u_A - u_B}{2} \leq v_A \leq \frac{u_A + u_B}{2}$

Solution

Method 1:

By momentum conservation

$$v_A + v_B = u_A + u_B$$

Consider energy:

$$v_A^2 + v_B^2 \leq u_A^2 + u_B^2$$

We should also have

$$v_A \leq v_B$$

because a tennis ball cannot "penetrate" another.

The three conditions are shown graphically by plotting v_B versus v_A.
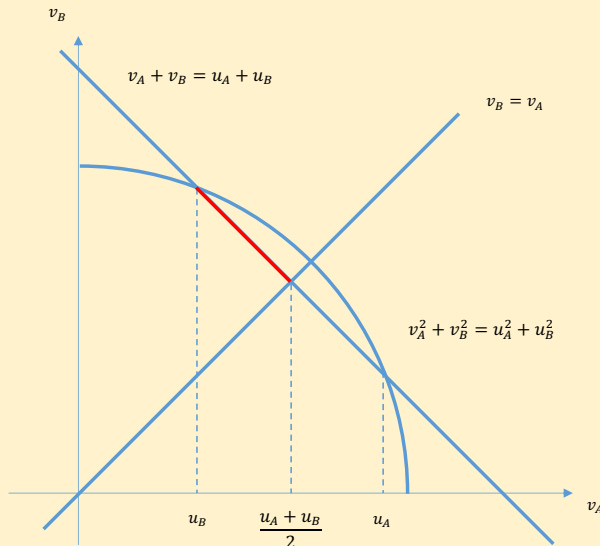
All possible (v_A, v_B) should lie on the straight line $v_A + v_B = u_A + u_B$, inside or on the circle $v_A^2 + v_B^2 = u_A^2 + u_B^2$, and on or "above" the straight line $v_B = v_A$. That is, the red closed line segment.

Therefore

$$u_B \leq v_A \leq \frac{u_A + u_B}{2}$$

Method 2

Kinetic energy must not increase:

$$v_A^2 + v_B^2 \leq u_A^2 + u_B^2$$

By momentum conservation

$$v_A + v_B = u_A + u_B$$

$$v_B = u_A + u_B - v_A$$

$$v_A^2 + (u_A + u_B - v_A)^2 \leq u_A^2 + u_B^2$$

$$v_A^2 + v_A^2 - 2(u_A + u_B)v_A + (u_A + u_B)^2 \leq u_A^2 + u_B^2$$

$$2v_A^2 - 2(u_A + u_B)v_A + 2u_A u_B \leq 0$$

$$v_A^2 - (u_A + u_B)v_A + u_A u_B \leq 0$$

$$(v_A - u_A)(v_A - u_B) \leq 0$$

Hence

$$u_B \leq v_A \leq u_A$$

Because

$$v_B = u_A + u_B - v_A \geq v_A$$

$$\frac{u_A + u_B}{2} \geq v_A$$

So

$$u_B \leq v_A \leq \frac{u_A + u_B}{2}$$

Method 3

The velocity of the CM is

$$\frac{u_A + u_B}{2}$$

In CM frame, the respective initial velocities of *A* and *B* are

$$u'_A = u_A - \frac{u_A + u_B}{2} = \frac{u_A - u_B}{2}$$

$$u'_B = u_B - \frac{u_A + u_B}{2} = \frac{u_B - u_A}{2}$$

Let the respective final velocities of *A* and *B* in CM frame be v'_A and v'_B.

By momentum conservation

$$v'_A + v'_B = 0$$

Kinetic energy must not increase:

$$v'^2_A + v'^2_B \leq 2 \times \left(\frac{u_A - u_B}{2}\right)^2 = \frac{(u_A - u_B)^2}{2}$$

Therefore

$$0 \leq v'^2_A, v'^2_B \leq \frac{(u_A - u_B)^2}{4}$$

Also

$$v'_A \leq 0 \leq v'_B$$

Hence

$$-\frac{u_A - u_B}{2} \leq v'_A \leq 0$$

Back to lab frame

$$\textcolor{red}{u_B} \leq v_A \leq \frac{u_A + u_B}{2}$$

Answer is C.

Hong Kong Physics Olympiad 2022 – Paper 2

2022 年香港物理奧林匹克競賽 – 卷二

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18 September 2022

2022 年 9 月 18 日

Rules and Regulations 競賽規則

1. All questions are in bilingual versions. You can answer in Chinese or English, but only ONE language should be used throughout the paper.

   所有題目均為中英對照。你可選擇以中文或英文作答，惟全卷必須以單一語言作答。

2. Please write your 3-digit Contestant Number and English Name on the first page of the answer books.

   在答題簿的第一頁上，請填上你的 3 位數字參賽者號碼及英文姓名。

3. The open-ended problems are quite long. Please read the whole problem first before attempting to solve them. If there are parts that you cannot solve, you are allowed to treat the answer as a known answer to solve the other parts.

   開放式問答題較長，請將整題閱讀完後再著手解題。若某些部分不會做，也可把它們的答案當作已知來做其他部分。

The following symbols and constants are used throughout the examination paper unless otherwise specified:

除非特別注明，否則本卷將使用下列符號和常數：

Gravitational acceleration on Earth's surface 地球表面重力加速度	g	9.80 ms^{-2}
Gravitational constant 重力常數	G	6.67×10^{-11} m^{3}kg^{-1}s^{-2}

The following identities may be useful in the competition:

1.

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x, \qquad \text{when } |x| \ll 1$$

2.

$$\sin x \approx \tan x \approx x. \qquad \text{when } x \ll 1 \text{ is measured in radian.}$$
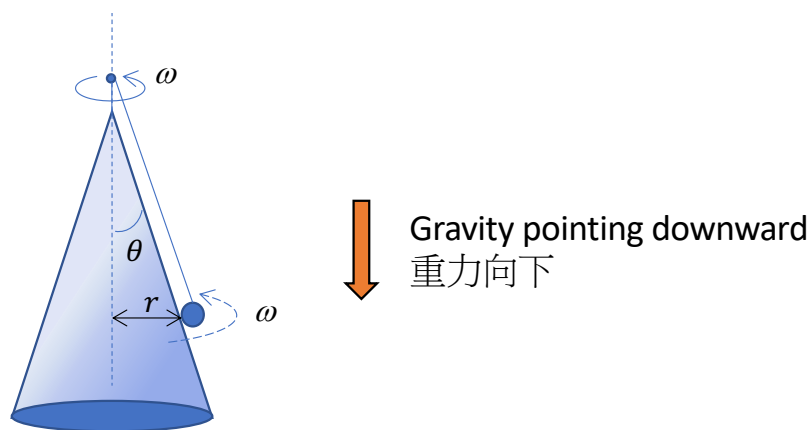
Open Problems 開放題

1. [15 pts] A bead of a mass m is hanged by a string from the tip of a cone, as shown in the figure below. The cone surface and the bead have a coefficient of static friction, $\mu_s = 0.7$ and a coefficient of kinetic friction, $\mu_k = 0.5$.

1. [15 pts] 如下圖所示，一個質量為 m 的珠子用一根細線從錐體的頂端懸掛下來。錐面和珠子間的靜摩擦系數和動摩擦系數分別為 $\mu_s = 0.7$和 $\mu_k = 0.5$。

The cone starts to rotate from rest and uniformly increases its angular speed to ω in τ seconds. The cone surface drags the bead to move at the same angular speed such that the bead has no relative motion to the cone surface (i.e. no sliding on the surface). After τ second, the cone is rotating at constant speed ω and the bead is moving in a uniform circular motion with a radius r at the same angular speed ω as shown in the figure below. Assume a massless string and neglect air friction. Answer the following questions in terms of m, g, ω, r, τ and θ.

圓錐體從靜止狀態開始旋轉，並在 τ 秒內將其角速度均勻增加到 ω。 錐面拖動珠子以相同的角速度移動，使得珠子與錐面沒有相對運動（即是在錐面上沒有滑動）。 τ 秒後，圓錐體以角速度 ω 旋轉，珠子以相同的角速度 ω 作半徑為 r 的勻速圓周運動，如下圖所示。 可假設細線沒有質量並忽略空氣摩擦。請用 m, g, ω, r, τ 和 θ 回答下列問題。

Gravity pointing downward
重力向下

In (a)-(c), we consider the case where the cone is rotating at constant speed ω.
在 (a)-(c) 中，我們考慮圓錐體以恆定速度 ω 旋轉的情況。

   (a) [1 pts] Find the linear speed of the bead, v, when the cone rotation is at ω.
      [1 pts] 求圓錐旋轉為 ω 時珠子的線性速度 v。

   (b) [2 pts] Draw the free body diagram of the bead indicating all the force vectors and the acceleration vector.
      [2 pts] 畫出珠子的自由體圖，表示所有的力和加速度的矢量。

   (c) [5 pts] Find the tension of the string, T, and the normal force, N, acting on the ball by the cone.
      [5 pts] 求細線的張力 T 和錐體作用在球上的法向力 N。

   (d) [2 pts] Find the linear acceleration of the bead along the direction of motion during the acceleration from $t = 0$ s to $t = \tau$ s.
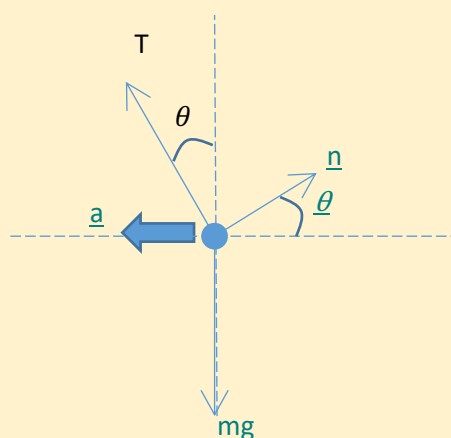   [2 pts] 求從 $t = 0$ s 到 $t = \tau$ s 的加速期間珠子沿運動方向的線性加速度。

(e) [5 pts] Find the maximum value of ω such that the bead will not slide on the cone surface throughout/at the end of the acceleration from $t = 0$ s to $t = \tau$ s. Given $m = 100$ gram, $g = 9.8$ m/s^2, $r = 1$ m, $\tau = 2$ and $\theta = 30°$.

[5 pts] 找出 ω 的最大值,使得在 $t = 0$ s 到 $t = \tau$ s 的整個加速過程中,珠子都不會在錐面上滑動。 給定 $m = 100$ 克,$g = 9.8$ m/s^2,$r = 1$ m,$\tau = 2$ 和 $\theta = 30°$。

Solution

(a) The linear speed is $v = r\omega$. [1pt]

(b)

[0.5 pt for each of the vector: mg, n, and T and a]

(c) From the free-body diagram, the Newton's 2nd law yields,

$$\begin{cases} T\cos\theta + n\sin\theta = mg. & [1.5] \\ T\sin\theta - n\cos\theta = mr\omega^2 & [1.5] \end{cases}$$

$$\Rightarrow \begin{cases} T = mg\cos\theta + mr\omega^2\sin\theta & [1] \\ n = mg\sin\theta - mr\omega^2\cos\theta & [1] \end{cases}$$

(d) $a_t = r\dfrac{\omega - 0}{\tau} = \dfrac{r\omega}{\tau}$ [2pt]

(e) The linear acceleration of the bead along its direction of motion equals to the friction on the bead by the cone. When the bead is not sliding, the friction is static. According to Newton's 2nd law,

$$\frac{mr\omega}{\tau} = F_{net} = \text{friction} \leq f_{s,max} = \mu_s n$$

[1 pt for knowing the tangential acceleration is due to friction]

[2 pt for knowing the friction is static and knowing it is bounded by the maximum value $\mu_s n$.]

When the required net force, F_{net}, excess the maximum friction, the bead will start to slide. Since the maximum friction decreased as the angular speed of the bead increases, the bead is most likely to slide at the end of the acceleration where it angular speed is ω. We have the condition,

$$\frac{mr\omega}{\tau} \leq \mu_s n = \mu_s mg \sin\theta - \mu_s mr\omega^2 \cos\theta$$

$$\Longrightarrow \frac{r\omega}{\tau} \leq \mu_s g \sin\theta - \mu_s r\omega^2 \cos\theta$$

$$\Longrightarrow \frac{r\omega}{\tau} \leq \mu_s g \sin\theta - \mu_s r\omega^2 \cos\theta$$

$$\Longrightarrow 0.35\sqrt{3}\,\omega^2 + 0.5\omega - 3.43 \leq 0$$

$$\Longrightarrow -2.83 \leq \omega \leq 2.00$$

Therefore, the maximum value is $\omega_{\text{max}} = 2.00$ rad/s.

[1 pt for solving the quadratic equations]

[1 pt for selecting the right answer.]

2. [15 pts] The figure below shows a small bead with mass m located on the surface of a semi-circular block with mass M and radius R. The surface of the block is smooth and there are no frictional forces between the two objects. The bead is initially put at the top of the block ($\theta = 0°$) and is given a very small initial push to the right so that it starts to slide down the block with negligible initial speed.

2. [15 pts] 下圖顯示了一個質量為 m 的珠子位於一個質量為 M 且半徑為 R 的半圓形板的表面上。板的表面光滑，兩個物體之間沒有摩擦力。珠子最初被放置在板的頂部（$\theta = 0°$），並被賦予一個非常小的向右的初始推動力，使其開始以可忽略不計的初始速度滑下。
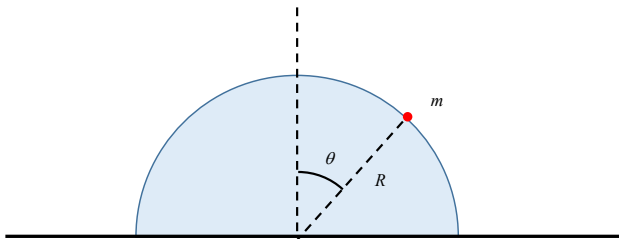
(a) If the block is fixed on the ground.
(a) 如果板固定在地面上。

    (ai)   [2pts] Draw the free-body diagram of the bead indicating all the force vectors and the acceleration vector.
            [2pts] 畫出珠子的自由體圖，表示所有的力矢量和加速度的矢量。

    (aii)   [4pts] Find the angle θ at which the bead will leave the surface of the block.
            [4pts] 求珠子離開板表面的角度 θ。

    (aiii)   [4pts] Find the distance of the bead from the center of the block when it hits the ground.
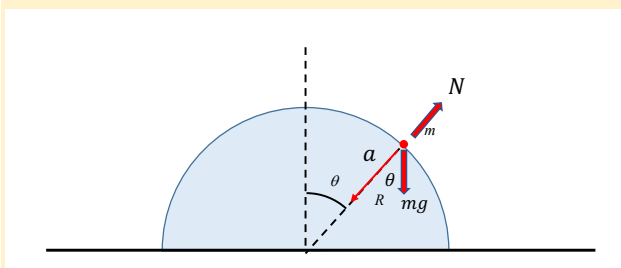   [4pts] 求珠子落地時距離板中心的距離。

(b) [5pts] If the block is free to slide on the ground under no friction and given that the bead does not leave the surface of the block, find the speed of the bead at angle θ.
(b) [5pts] 如果板在沒有摩擦力的情況下在地面上自由移動，並且給珠子不會離開板的表面，求在角度 θ 時珠子的速率。

Solution:

Solution

(ai)

[0.5] for N; [0.5] for mg; [1] for a

(aii)   Let the normal reaction be N at angle θ. Then

$$mg\cos\theta - N = m\frac{v^2}{R} \quad [1]$$

By energy conservation

$$mgR(1-\cos\theta) = \frac{1}{2}mv^2 \quad [1]$$

So

$$mg\cos\theta - N = 2mg(1-\cos\theta)$$

When $N = 0$,

$$mg\cos\theta = 2mg(1-\cos\theta)$$

$$3\cos\theta = 2$$

$$\theta = \cos^{-1}\frac{2}{3} \quad [1+1]$$

(aiii)   Let $\theta = \cos^{-1}\frac{2}{3}$, $v = \sqrt{\frac{2}{3}gR}$ $\quad [0.5]$

$$x(t) = R\sin\theta + vt\cos\theta \quad [0.5]$$

$$y(t) = R\cos\theta - vt\sin\theta - \frac{1}{2}gt^2 \quad [0.5]$$

Solve for

$$R\cos\theta - vt\sin\theta - \frac{1}{2}gt^2 = 0. \ [0.5]$$

Get

$$t = \frac{v\sin\theta \pm \sqrt{v^2\sin^2\theta + 2gR\cos\theta}}{-g}$$

Since $t > 0$,

$$t = -\frac{v\sin\theta}{g} + \frac{\sqrt{v^2\sin^2\theta + 2gR\cos\theta}}{g} \quad [1]$$

$$= -\frac{\sqrt{\frac{2}{3}gR}\sqrt{\frac{5}{9}}}{g} + \frac{\sqrt{\frac{2}{3}gR\frac{5}{9} + 2gR\frac{2}{3}}}{g}$$

$$= \left(\sqrt{\frac{46}{27}} - \sqrt{\frac{10}{27}}\right)\sqrt{\frac{R}{g}} = \frac{\sqrt{46} - \sqrt{10}}{3\sqrt{3}}\sqrt{\frac{R}{g}}$$

$$x(t) = R\sqrt{\frac{5}{9}} + \sqrt{\frac{2}{3}gR}\left(\frac{\sqrt{46} - \sqrt{10}}{3\sqrt{3}}\sqrt{\frac{R}{g}}\right)\frac{2}{3}$$

$$= R\left[\sqrt{\frac{5}{9}} + \sqrt{\frac{2}{3}}\left(\frac{\sqrt{46}-\sqrt{10}}{3\sqrt{3}}\right)\frac{2}{3}\right]$$

$$= \left(\frac{\sqrt{5}}{3} + \frac{2(\sqrt{92}-\sqrt{20})}{27}\right)R$$

$$= \frac{9\sqrt{5} + 4\sqrt{23} - 4\sqrt{5}}{27}R$$

$$= \frac{5\sqrt{5} + 4\sqrt{23}}{27}R = 1.125R \quad \text{[1]}$$

(b)   Let the horizontal velocity of M be V (right = +ve) and the speed of m relative to M be v. Then the horizontal velocity of m is $V + v\cos\theta$ and the downward vertical velocity is $v\sin\theta$. By momentum conservation in horizontal direction

$$MV + m(V + v\cos\theta) = 0. \quad \text{[1]}$$

$$V = -\frac{mv\cos\theta}{M+m}$$

By energy conservation

$$mgR(1-\cos\theta) = \frac{1}{2}MV^2 + \frac{1}{2}m(V + v\cos\theta)^2 + \frac{1}{2}mv^2\sin^2\theta. \quad \text{[1]}$$

$$mgR(1-\cos\theta) = \frac{1}{2}M\left(-\frac{mv\cos\theta}{M+m}\right)^2 + \frac{1}{2}m\left(-\frac{mv\cos\theta}{M+m} + v\cos\theta\right)^2 + \frac{1}{2}mv^2\sin^2\theta$$

$$2mgR(1-\cos\theta) = \frac{Mm^2}{(M+m)^2}v^2\cos^2\theta + m\left(-\frac{m}{M+m} + 1\right)^2 v^2\cos^2\theta + mv^2\sin^2\theta$$

$$2mgR(1-\cos\theta) = \frac{Mm^2}{(M+m)^2}v^2\cos^2\theta + \frac{M^2m}{(M+m)^2}v^2\cos^2\theta + mv^2\sin^2\theta$$

$$2mgR(1-\cos\theta) = \frac{Mm}{M+m}v^2\cos^2\theta + mv^2\sin^2\theta$$

$$2gR(1-\cos\theta) = \frac{M}{M+m}v^2(1-\sin^2\theta) + v^2\sin^2\theta$$

$$2gR(1-\cos\theta) = \frac{M}{M+m}v^2 + \frac{m}{M+m}v^2\sin^2\theta$$

$$2(M+m)gR(1-\cos\theta) = (M + m\sin^2\theta)v^2$$

([2] for the step to solve the equations)

$$v^2 = 2gR\frac{(M+m)(1-\cos\theta)}{M+m\sin^2\theta}$$

$$v = \sqrt{2gR\frac{(M+m)(1-\cos\theta)}{M+m\sin^2\theta}} \quad \text{[1]}$$

3. [20 pts] Newton's law of gravitation holds only for two point objects. Nevertheless, for finite-sized objects with spherically symmetric mass distributions, the gravitational forces are still simple due to the following "Shell theorem":

The gravitational force by a thin (negligible thickness) spherical shell with uniform mass density exerting on a small test mass

> (i) is the same as if all of its mass were concentrated at a point at its center, if the test mass is outside the shell, and
> (ii) is zero (no net gravitational force), if the test mass is inside the shell.

3. [20 pts] 牛頓萬有引 陌質量分布的物體, 基於"殼層定理", 其引力作用依然簡單。

殼層定理:
一個質量均勻分布的 質量的引力有以下性質:

> (i) 如試驗質量位於殼層外, 引力與假想所有殼層質量皆位於球心而給出的一樣。
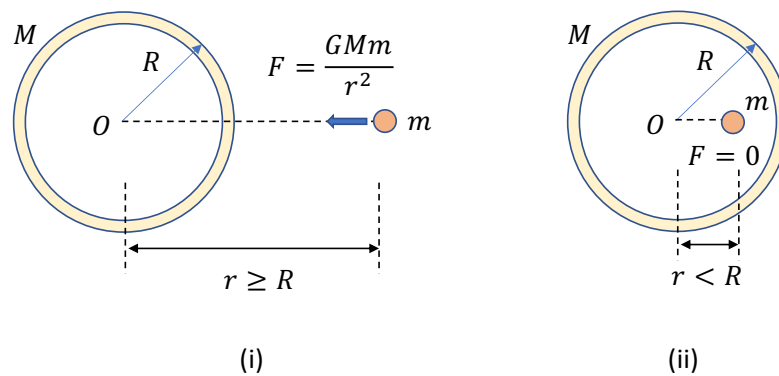> (ii) 如試驗質量位於殼層內, 其受到的淨引力為零。

Figure 3a: Shell theorem for (i) $r \geq R$ and (ii) $r < R$ respectively.

圖 3a: 分別為 (i) $r \geq R$ 和 (ii) $r < R$ 的殼層定理。

(a) [3pts] For a uniform solid sphere with radius R and mass M, let its center be the origin and denote the position vector of the observation point by $\vec{r}$. Find the gravitational force $\vec{F}(\vec{r})$ acting on a point mass m at $\vec{r}$ for $r \geq R$ and $r < R$ respectively. Express your answer in terms of $\vec{r}$, m, and ρ, where ρ is the uniform mass density of the solid sphere given by

(a) [3pts] 對於一個半徑為 R、質量為 M 的均勻實心球體，以它的中心為原點，用 $\vec{r}$ 表示觀察點的位置向量。 求作用在 $\vec{r}$ 處點質量 m 上的重力 $\vec{F}(\vec{r})$，分別考慮 $r \geq R$ 和 $r < R$ 的情況。 用 $\vec{r}$、m 和 ρ 表達你的答案，其中 ρ 是由下式給出的實心球體的均勻質量密度

$$\rho = \frac{M}{4\pi R^3/3}.$$

An alien civilization with very advanced technology in astronomical engineering created an artificial planet with a hole in it, as shown in the figure below, so that they can reside inside the hole because it is too cold outside. The planet is made of material with uniform density ρ. The planet's surface and the hole are both spherical, with

radius R and $R/2$, respectively, so that the hole touches the center of the planet.

一個天體工程技術非常先進的外星文明創造了一個人造星球，其中有一個洞，如下圖所示，因為外面太冷，他們可以住在洞裡。 行星由均勻密度為 ρ 的材料構成。 行星表面和洞都是球形的，半徑分別為 R 和 $R/2$，因此洞可接觸行星的中心。
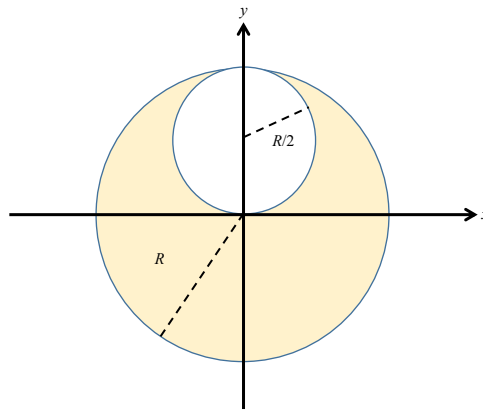
(b) [5pts] Show that the gravitational force acting on a point mass m inside the hole is given by $\vec{F} = -mg'\hat{j}$ for some positive constant g' and $\hat{j}$ is the unit vector along the y-axis. Find g' in terms of ρ and R.

(b) [5pts] 證明作用在洞內點質量 m 上的重力由 $\vec{F} = -mg'\hat{j}$ 給出，其中 g' 是某個正常數，$\hat{j}$ 是沿 y 軸的單位向量。求 g'，用 ρ 和 R 表示。

If you cannot solve part (b), you may still use the result of part (b) to answer the following parts and express your answers in terms of g'.

如果你不能解答（b）部分，你仍可使用（b）部分的結果來回答以下問題，並用 g' 表示你的答案。

(c) [4pts] A small object with mass m is launched from the center of the hole in the positive x direction with initial speed u. Find the y coordinate of the point at which the object hits the surface of the hole.

[4pts] 一個質量為 m 的小物體以初始速率 u 從洞中心向正 x 方向發射。 求物體撞擊洞表面處的 y 坐標。

(d) [4pts] If the object can be launched from the center of the hole at any angle, then it will be able to hit any point of the surface of the hole when $u \geq u_c$ for some critical speed u_c. If $u < u_c$, then the object cannot hit the surface of the hole at which $y > y_{max}$ for some y_{max}. Find u_c and y_{max}.

[4pts] 如果物體可以從洞中心以任意角度發射，那麼當 u 大於某個臨界速率 u_c 時 $(u \geq u_c)$，它能撞擊到洞表面的任意一點。如果 $u < u_c$，則物體不能撞擊高於某個臨界高度 y_{max} 處的洞表面。求 u_c 及 y_{max}。

(e) [4pts] If the object is launched from the center of the hole in the positive y direction and assumes that there is a very small opening at the top of the hole, find the minimum launching speed so that the object can escape to infinity.
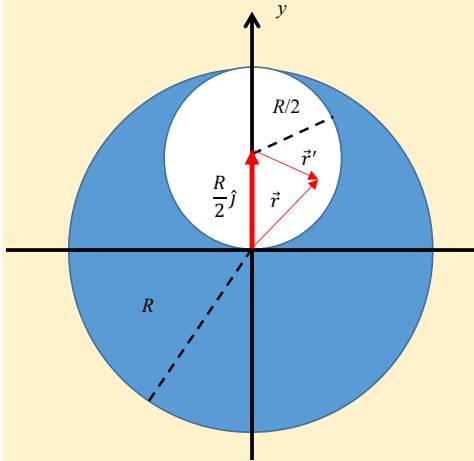
[4pts] 如果物體從洞中心向正 y 方向發射，並假設孔的頂部有一個很小的開口，求最小發射速率，使物體可以逃到無窮遠。

Solution

(a)

$$\vec{F}(\vec{r}) = \begin{cases} -\dfrac{G\rho\left(\frac{4\pi R^3}{3}\right)m}{r^2}\hat{r} = -\dfrac{4\pi G\rho R^3 m}{3r^2}\hat{r} = -\dfrac{4\pi G\rho R^3 m}{3r^3}\vec{r} & \text{for } r \geq R. \; [1] \\[4mm] -\dfrac{G\rho\left(\frac{4\pi r^3}{3}\right)m}{r^2}\hat{r} = -\dfrac{4\pi G\rho m}{3}r\hat{r} = -\dfrac{4\pi G\rho m}{3}\vec{r} & \text{for } r < R. \; [2] \end{cases}$$

(b)

$$\vec{F} = -Gm\left(\rho\frac{4}{3}\pi r^3\right)\frac{\vec{r}}{r^3} - Gm\left(-\rho\frac{4}{3}\pi r'^3\right)\frac{\vec{r}'}{r'^3} \quad [1+1]$$

$$= -Gm\left(\rho\frac{4}{3}\pi\right)\vec{r} + Gm\left(\rho\frac{4}{3}\pi\right)\vec{r}' \quad [1]$$

$$= -Gm\left(\rho\frac{4}{3}\pi\right)(\vec{r} - \vec{r}') \quad [1]$$

$$= -Gm\left(\rho\frac{4}{3}\pi\right)\frac{R}{2}\hat{j}$$

$$= -m\left(\frac{2}{3}\pi G\rho R\right)\hat{j}$$

$$= -mg'\hat{j}$$

$$g' = \frac{2}{3}\pi G\rho R. \quad [1]$$

(c)

$$x(t) = ut.$$

$$y(t) = \frac{R}{2} - \frac{1}{2}g't^2$$

$$y = \frac{R}{2} - \frac{1}{2}g'\left(\frac{x}{u}\right)^2$$

$$y = \frac{R}{2} - \frac{g'}{2u^2}x^2 \quad [1]$$

$$x^2 + \left(y - \frac{R}{2}\right)^2 = \frac{R^2}{4} \quad [1]$$

$$\left(y - \frac{R}{2}\right)^2 - \frac{2u^2}{g'}\left(y - \frac{R}{2}\right) - \frac{R^2}{4} = 0$$

$$y - \frac{R}{2} = \frac{1}{2}\left[\frac{2u^2}{g'} \pm \sqrt{\frac{4u^4}{g'^2} + R^2}\right]$$

$$y = \frac{R}{2} + \frac{u^2}{g'} \pm \sqrt{\frac{u^4}{g'^2} + \frac{R^2}{4}} \quad [1]$$

Since $y \le R$,

$$y = \frac{R}{2} + \frac{u^2}{g'} - \sqrt{\frac{u^4}{g'^2} + \frac{R^2}{4}}$$

$$y = \frac{R}{2} + \frac{3u^2}{2\pi G\rho R} - \sqrt{\left(\frac{3u^2}{2\pi G\rho R}\right)^2 + \frac{R^2}{4}} \quad [1]$$

(d)   It is clear that it is always possible to hit (0, 0).

To hit (0, R), launch upwards with speed $u \ge \sqrt{g'R}$.

Now for (x, y) on the hole surface where $x \ne 0$ and hence $0 < y < R$,

$$x(t) = ut\cos\theta$$

$$y(t) = \frac{R}{2} + ut\sin\theta - \frac{1}{2}g't^2$$

$$y = \frac{R}{2} + x\tan\theta - \frac{g'\sec^2\theta}{2u^2}x^2$$

$$y = \frac{R}{2} + x\tan\theta - \frac{g'x^2}{2u^2}(1 + \tan^2\theta)$$

$$0 = \frac{R}{2} - y - \frac{g'x^2}{2u^2} + x\tan\theta - \frac{g'x^2}{2u^2}\tan^2\theta$$

There will be no solution for the above quadratic equation in $\tan\theta$ if

$$x^2 - 4\left(-\frac{g'x^2}{2u^2}\right)\left(\frac{R}{2} - y - \frac{g'x^2}{2u^2}\right) < 0. \quad [1]$$

$$1 + \frac{2g'}{u^2}\left(\frac{R}{2} - y - \frac{g'x^2}{2u^2}\right) < 0$$

$$1 < \frac{2g'}{u^2}\left(y - \frac{R}{2} + \frac{g'}{2u^2}\left(\frac{R^2}{4} - \left(y - \frac{R}{2}\right)^2\right)\right)$$

$$\frac{u^2}{2g'} < \frac{g'R^2}{8u^2} + y - \frac{R}{2} - \frac{g'}{2u^2}\left(y - \frac{R}{2}\right)^2$$

$$\frac{u^4}{g'^2} < \frac{R^2}{4} + \frac{2u^2}{g'}\left(y - \frac{R}{2}\right) - \left(y - \frac{R}{2}\right)^2$$

$$0 > \left(y - \frac{R}{2}\right)^2 - \frac{2u^2}{g'}\left(y - \frac{R}{2}\right) - \frac{R^2}{4} + \frac{u^4}{g'^2}$$

$$\frac{R^2}{4} > \left[\left(y - \frac{R}{2}\right) - \frac{u^2}{g'}\right]^2$$

$$-\frac{R}{2} < y - \frac{R}{2} - \frac{u^2}{g'} < \frac{R}{2}$$

$$\frac{u^2}{g'} < y < R + \frac{u^2}{g'} \quad [1]$$

Since $0 < y < R$, the only condition is

$$\frac{u^2}{g'} < y$$

When

$$\frac{u^2}{g'} \geq R \Longleftrightarrow u \geq \sqrt{g'R}$$

the condition can never be satisfied and therefore the projectile can hit all points on the hole surface with $x \neq 0$.

We also know that it can always hit (0, 0) and in this case it can hit (0, R).

Therefore

$$u_c = \sqrt{g'R} = \sqrt{\frac{2}{3}\pi G\rho R^2} \quad [1]$$

When $u < u_c$

$$y_{max} = \frac{u^2}{g'} = \frac{3u^2}{2\pi G\rho R} < R. \quad [1]$$

(e)   By energy conservation

$$\frac{1}{2}mu^2 = m\left(\frac{2}{3}\pi G\rho R\right)\frac{R}{2} + \frac{Gm\left(\rho\frac{4}{3}\pi R^3\right)}{R} + \frac{Gm\left(-\rho\frac{4}{3}\pi\left(\frac{R}{2}\right)^3\right)}{R/2} \quad [1+1]$$

$$\frac{1}{2}u^2 = \frac{1}{3}\pi G\rho R^2 + \frac{4}{3}\pi G\rho R^2 - \frac{1}{3}\pi G\rho R^2 \quad [1]$$

$$u = \sqrt{\frac{8}{3}\pi G\rho R^2} \quad [1]$$