Hong Kong Physics Olympiad Lesson 1

1.1 Dimensional Analysis

The fundamental quantities used in physical descriptions are called dimensions. Length, mass, and time are examples of dimensions. You could measure the distance between two points and express it in units of meters, centimeters, or feet. In any case, the quantity would have the dimension of length.

It is common to express dimensional quantities by bracketed symbols, such as [L], [M], and [T] for length, mass, and time, respectively. Dimensional analysis is a procedure by which the dimensional consistency of any equation may be checked. If I say, $x = at$, where x, a, and t represent the displacement, acceleration and time respectively. Is it correct?

- $[x] = [\mathrm{L}]$
- $[at] = ([\mathrm{L}][\mathrm{T}]^{-2}) ([\mathrm{T}]) = [\mathrm{L}][\mathrm{T}]^{-1}$

So the dimension for the sides are not equal, i.e. $x = at$ is invalid.

Now look at the equation $x = at^2$

- The L.H.S. is x, i.e. [L]
- The R.H.S. is at^2, i.e. $([\mathrm{L}][\mathrm{T}]^{-2}) ([\mathrm{T}]^2) = [\mathrm{L}]$,

Hence the equation is dimensionally correct, but you should know that it is physically incorrect. The correct equation is, as you should remember, $x = \frac{1}{2}at^2$. The fraction $\frac{1}{2}$ is a constant and has no dimension, just like π.

Example

A planet moving in a circular orbit experiences acceleration. This acceleration depends only on the speed v of the planet and on the radius r of its orbit. How can we use dimensional analysis to determine the acceleration, a, and relate it to v and r?

Answer:

We assume that the acceleration is proportional to some power of the speed, e.g. $a \propto v^p$. Similarly, we assume that the acceleration is proportional to some power of the orbit radius, e.g. $a \propto r^q$. Our assumed form for the acceleration can be expressed by

$$a = v^p r^q.$$

The exponents p and q are unknown numbers that we are going to find out by using dimensional analysis.

- $[a]$=[L][T]$^{-2}$
- $[v]$=[L][T]$^{-1}$
- $[r]$=[L]

Hence the L.H.S. of the assumed form is [L][T]$^{-2}$, and that for the R.H.S. is [L]$^{p+q}$[T]$^{-p}$. Equate the index of [L] and [T], we have two equations

$$-2 = -p\ ,$$

$$1 = p+q\ .$$

After solving, we obtain $p = 2$ and $q = -1$. Accordingly, we can write a dimensionally correct relation $a = \dfrac{v^2}{r}$.

Example

The period P of a simple pendulum is the time for one complete swing. How does P depend on the mass m of the bob, the length l of the string, and the acceleration due to gravity g?

Answer:

We begin by expressing the period P in terms of the other quantities as follows:

$$P = k\, m^x\, l^y\, g^z$$

where k is a constant and x, y, and z are to be determined. Next we insert the dimensions of each quantity:

$$P = (\mathrm{M}^x)(\mathrm{L}^y)(\mathrm{L}^z\mathrm{T}^{-2z})$$

$$P = (\mathrm{M}^{x})(\mathrm{L}^{y+z})(\mathrm{T}^{-2z})$$

And equate the powers of each dimension of either side of the equation. Thus,

$$\mathrm{T}: \quad 1 = -2z$$

$$\mathrm{M}: \quad 0 = x$$

$$\mathrm{L}: \quad 0 = y + z$$

These equations are easily solved and yield $x = 0$, $z = -\frac{1}{2}$, and $y = \frac{1}{2}$. Thus,

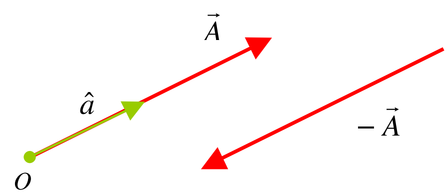
$$P = k\sqrt{\frac{l}{g}}.$$

1.2 Vectors

(a) *Definition of a vector*

$\boldsymbol{A} = A\hat{a}$

A: magnitude $\hat{a}$: unit vector with direction pointed



(b) *Cartesian coordinate system*

- 2-D (i.e. a plane)

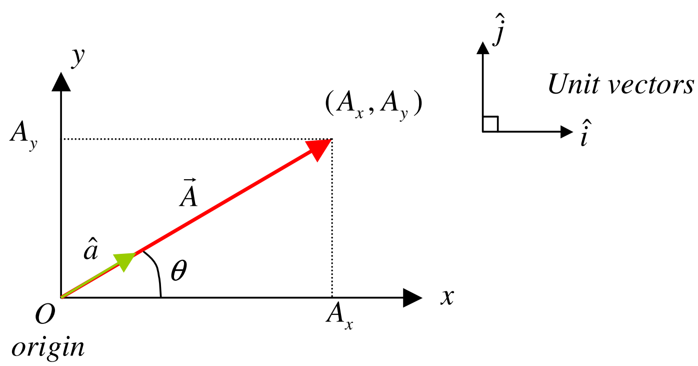
$\boldsymbol{A} = A_x\hat{i} + A_y\hat{j}$

A_x: Component of $\boldsymbol{A}$ in x-direction

A_y: Component of $\boldsymbol{A}$ in y-direction

The magnitude of $\boldsymbol{A}$ is given by

$$A = |A| = \sqrt{A_x^2 + A_y^2}.$$



Note that we have the components of $\boldsymbol{A}$

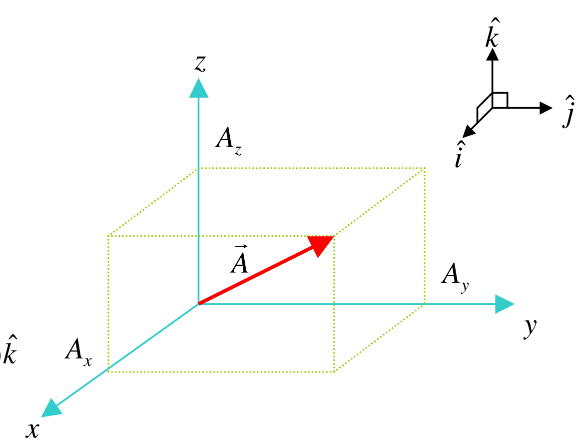
$A_x = A\cos\theta$ and $A_y = A\sin\theta$

- 3-D

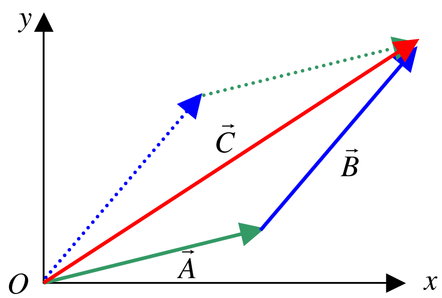
$\boldsymbol{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\boldsymbol{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

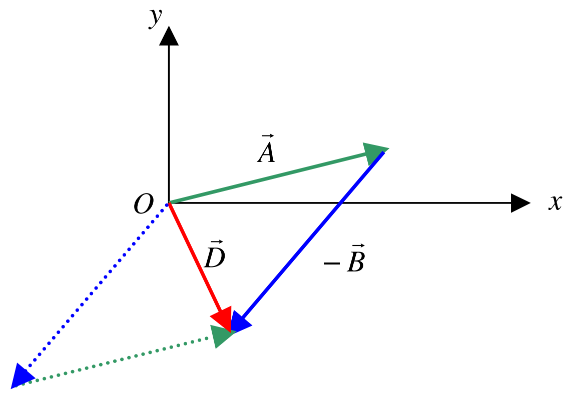
$\Rightarrow \boldsymbol{A} + \boldsymbol{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$



(c) *Mathematics of vectors*

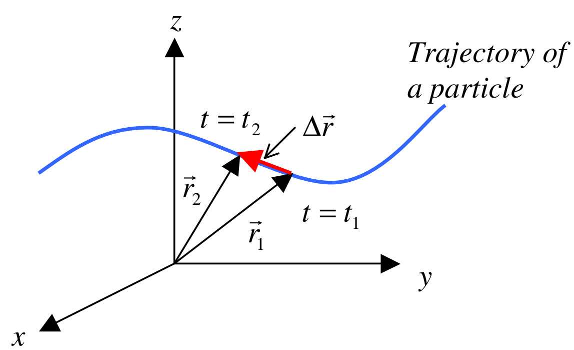


C = A + B



D = A − B

(d) *Examples of vectors*



(1) Position vector

$$\boldsymbol{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

(2) Displacement (change in position)

$$\Delta\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1 = \boldsymbol{r}(t_2) - \boldsymbol{r}(t_1)$$

(3) Average velocity during time Δt

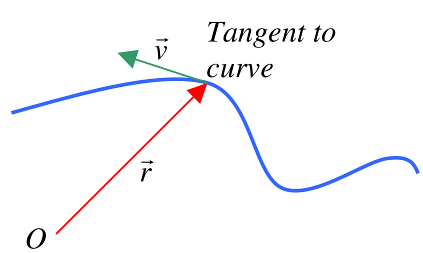
where $\Delta t = t_2 - t_1$

Velocity : A vector !

Note the difference of velocity and speed: speed is a scalar!

$$\boldsymbol{v}_{av} = \frac{\Delta\boldsymbol{r}}{\Delta t} = \frac{\boldsymbol{r}(t_1 + \Delta t) - \boldsymbol{r}(t_1)}{\Delta t}$$

(4) Instantaneous velocity



$$v \equiv \lim_{\Delta t \to 0} \frac{\Delta\boldsymbol{r}}{\Delta t} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

$$\lim_{\Delta t \to 0} \frac{\Delta\boldsymbol{r}}{\Delta t} = \frac{d\boldsymbol{r}}{dt}$$

(5) Acceleration

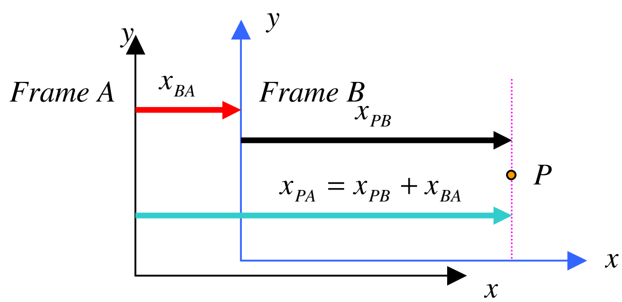
$$\boldsymbol{a}_{av} = \frac{\Delta\boldsymbol{v}}{\Delta t} = \frac{\boldsymbol{v}(t_1 + \Delta t) - \boldsymbol{v}(t_1)}{\Delta t}$$

$$\boldsymbol{a} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2\boldsymbol{r}}{dt^2} \leftarrow \textit{second derivative}$$

Remark: As we have $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{a}t$, and then we can write $\frac{d\boldsymbol{v}}{dt} = \boldsymbol{a}$.

1.3 Relative motion in one dimension

Each observer – such as you standing on the ground - defines a reference frame. The reference frame requires a coordinate system and a set of clocks, which enable an observer to measure positions, velocities, and accelerations in his or her particular frame.



As shown in the figure, we have two different frames to watch an object *P*.

Obviously, $x_{PA} = x_{PB} + x_{BA}$

In words:

" *The position of P as measured by frame A is equal to the position of P as measured by frame B plus the position B as measured by A*."

As $v = \frac{dx}{dt}$, we take the time derivative x_{PA} and obtain the relation of velocities.

$$v_{PA} = v_{PB} + v_{BA}$$

If two frames move at constant speed with respect to each other, *i.e.* $v_{BA} = const.$, take $\frac{d}{dt}$ again, we have $a_{PA} = a_{PB}$ (if $\frac{dv_{BA}}{dt} = 0$)

In general, $\boldsymbol{a}_{PA} = \boldsymbol{a}_{PB} + \boldsymbol{a}_{BA}$

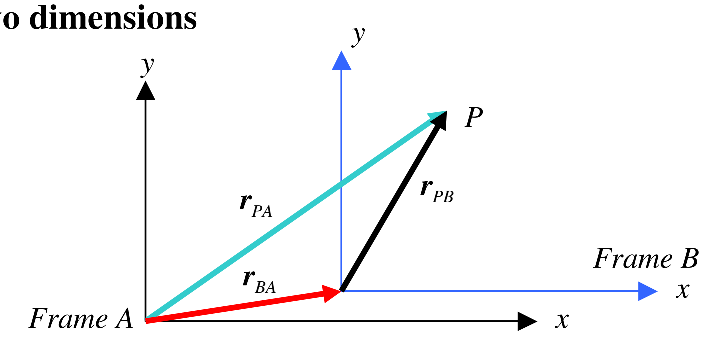
1.4 Relative motion in two dimensions

We have

$$\boldsymbol{r}_{PA} = \boldsymbol{r}_{PB} + \boldsymbol{r}_{BA}$$

and

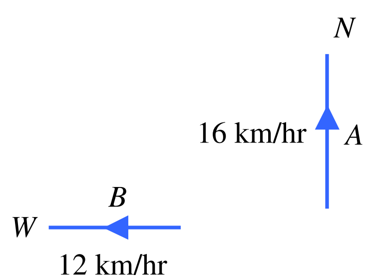
$$\boldsymbol{v}_{PA} = \boldsymbol{v}_{PB} + \boldsymbol{v}_{BA}$$

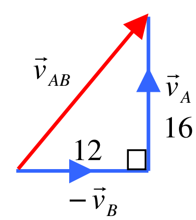


Example

A ship *A* is steaming due north at 16 km/hr and a ship *B* is steaming due west at 12 km/hr. Find the relative velocity of *A* with respect to *B*.

Answer:

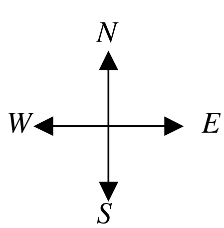


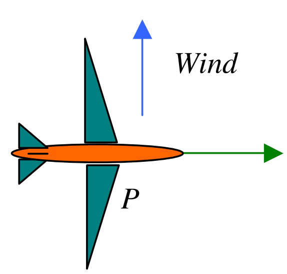


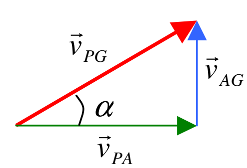
The relative velocity of *A* with respect to *B* is $\vec{v}_{AB}$ which equals to $\vec{v}_A - \vec{v}_B$. Referring to the vector diagram, the relative velocity has magnitude $|\vec{v}_{AB}| = \sqrt{12^2 + 16^2} = 20$. The direction of $\vec{v}_{AB}$ is $\mathrm{N}\tan^{-1}(\frac{12}{16})$ E, i.e. N 36°52' E.

Example

Suppose we have an airplane, the compass in the plane indicates that it is headed due east and is moving through air with speed 215 km/h. A steady wind of 65 km/h is blowing due north. What is the velocity of the plane with respect to the ground?







Answer:

$$\boldsymbol{v}_{PG} = \boldsymbol{v}_{PA} + \boldsymbol{v}_{AG}$$

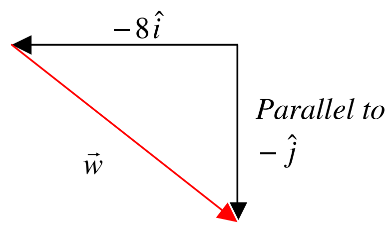
$$v_{PG} = \sqrt{v_{PA}^2 + v_{AG}^2} = \sqrt{215^2 + 65^2} \cong 225\, km/h > \text{Air speed}$$

$$\alpha = \tan^{-1}(\frac{v_{AG}}{v_{PA}}) = \tan^{-1}(\frac{65}{215}) = 16.8^\circ$$

Example

A man traveling East at 8 km h^{-1} finds that the wind seems to blow directly from the North. On doubling his speed he finds that it appears to come *NE*. Find the velocity of the wind.

Answer:



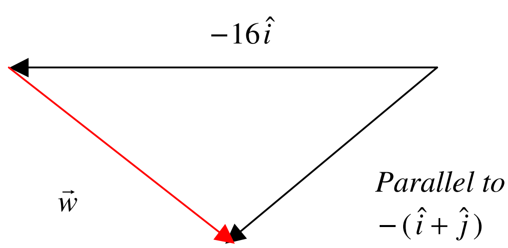
Let the velocity of the wind be

$$\vec{w} = x\hat{i} + y\,\hat{j}\,.$$

Then the velocity of the wind relative to the man is

$$\vec{w} - 8\hat{i} = (x-8)\,\hat{i} + y\,\hat{j}$$

But this is from the North, and is therefore parallel to $-\hat{j}$. Hence, we obtain $x - 8 = 0$.



When the man doubles his speed, the velocity of the wind relative to him is

$$\vec{w} - 16\hat{i} = (x-16)\,\hat{i} + y\,\hat{j}\,.$$

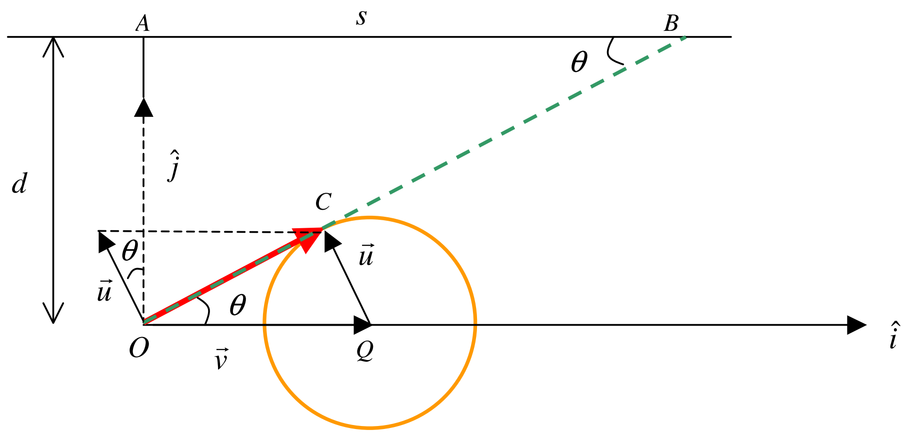
But this is from the *NE* and is therefore parallel to $-(\hat{i} + \hat{j})$. Hence $y = x - 16 = -8$.

The velocity of the wind is $8\,\hat{i} - 8\,\hat{j}$, which is equivalent to $8\sqrt{2}$ km h^{-1} from *NW*.

Example

A swimmer wishes to across a swift, straight river of width d. If the speed of the swimmer in still water is u and that of the water is v, where $v > u$, what is the direction along which the swimmer should proceed such that the downstream distance he has traveled when he reaches the opposite bank is the smallest possible? What is the corresponding time required?

Answer:



The downstream distance is smallest when OB is tangent to the circle, centered Q, radius u.

$$\sin\theta = \frac{u}{v} \text{ , and } AB = d\cot\theta = \frac{d\cos\theta}{\sin\theta} = \frac{d\sqrt{1-(\frac{u}{v})^2}}{\frac{u}{v}} = \frac{d\sqrt{v^2-u^2}}{u}.$$

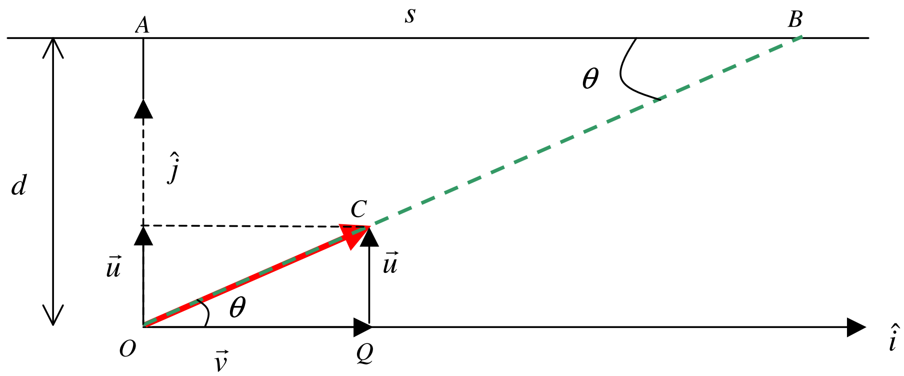
The time taken is

$$t = \frac{d}{u\cos\theta} = \frac{d}{u\sqrt{1-(\frac{u}{v})^2}} = \frac{dv}{u\sqrt{v^2-u^2}}.$$

Example

As an extension of the last example, if the swimmer wants to cross the swift with a minimum time, find the minimum time and the position that he reaches in the opposite bank.

Answer:



The minimum time of travel can be obtained if the swimmer directs in the direction of $\hat{j}$. The magnitude of $\vec{u}$ is contributed completely in the direction of crossing the swift. It is the fastest way. Hence the minimum time $t = d/u$. The distance downstream $s = vt = (vd)/u$.

1.5 Motion with constant acceleration a (1-D)

- As a is a constant and by definition $a = a_{av} = \dfrac{v(t) - v(0)}{t - 0}$.

  Hence, we have

  $$\boxed{v = u + at} \qquad (1)$$

  where $u = v(0)$.

- From (1), $v = \dfrac{dx}{dt} = u + at$

  $$\therefore \quad \boxed{x = x_0 + ut + \tfrac{1}{2}at^2} \qquad (2)$$

- It is quite often to have a reference point, e.g. $x_0 = 0$.

  Differentiate (2) with respect to time t, we obtain (1), e.g.

  $$\frac{dx}{dt} = \frac{dx_0}{dt} + \frac{d(ut)}{dt} + \frac{d(\frac{1}{2}at^2)}{dt}. \qquad \because \frac{dx}{dt} = v,\ \frac{dx_0}{dt} = 0,\ \frac{d(ut)}{dt} = u \text{ and } \frac{d(\frac{1}{2}at^2)}{dt} = at.$$

- $(1)^2 \Rightarrow v^2 = u^2 + 2aut + a^2t^2 = u^2 + 2a(ut + \tfrac{1}{2}at^2)$

$\therefore$ $\boxed{v^2 = u^2 + 2a(x - x_0)}$

- From (2), $x = x_0 + ut + \frac{1}{2}at^2 = x_0 + \frac{1}{2}t(at + u + u)$

  $\therefore$ $\boxed{x = x_0 + \frac{1}{2}(v + u)t}$

Remark #1:

It is quite often to read the set of equations in the textbook in secondary school as

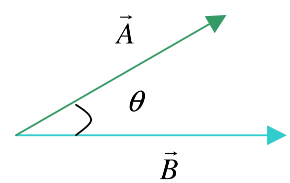
$$v = u + at\ ,$$

$$S = ut + \frac{1}{2}at^2\ ,$$

and $$v^2 = u^2 + 2aS\ .$$

Remark #2:

Dot product of two vectors, e.g. $\boldsymbol{A}$ and $\boldsymbol{B}$



$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{B} &\equiv A_x B_x + A_y B_y + A_z B_z \\ &= AB\cos\theta \end{aligned}$$

Example

A police officer is chasing a burglar across a rooftop, both are running at 4.5m/s. Before the burglar reaches the edge of the roof, he has to decide whether or not to try jumping to the roof of the next building, which is 6.2m away but 4.8m lower. Can he make it? Assume that $v_{0y} = 0$.

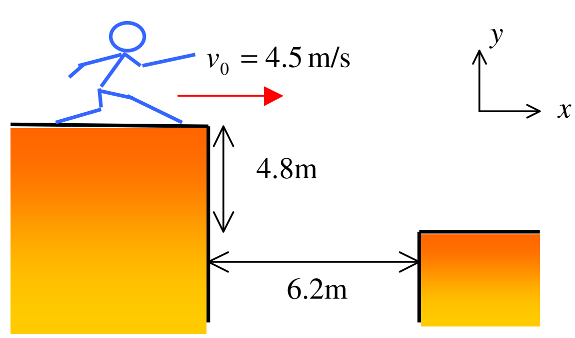
Answer:

- Note that he has to fall 4.8m and at the same time he need to moves at least 6.2m horizontally.

  $y = -4.8$ m

  $\because$ $y = v_{0y}t - \frac{1}{2}gt^2$ $(v_{0y} = 0)$

  $\therefore t = \sqrt{\dfrac{2(4.8)}{g}} = 0.99s$



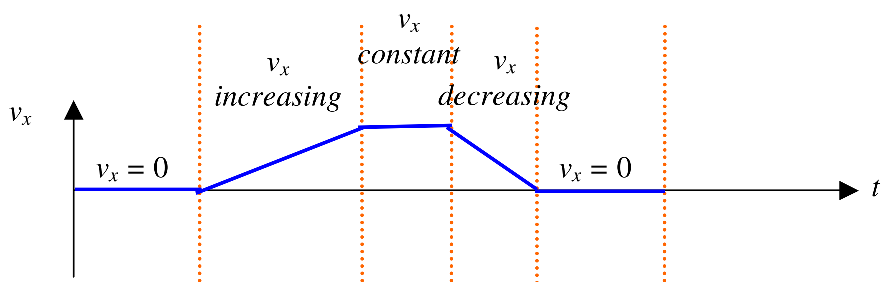
- How far can he move along x-direction?

$$x = v_{0x}t = 0.45(0.99) \approx 4.5m < 6.2m$$

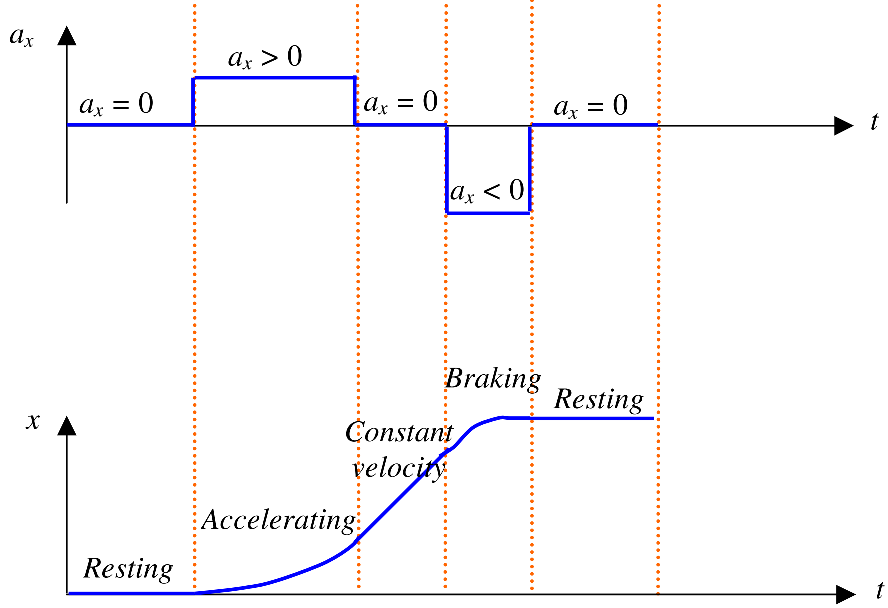
$\therefore$ Our advice: Don't jump!

Example

A car is at rest at a reference point. If the car moves in 1-D and it's motion is governed by the velocity-time graph given below, sketch the acceleration-time graph and the displacement-time graph.

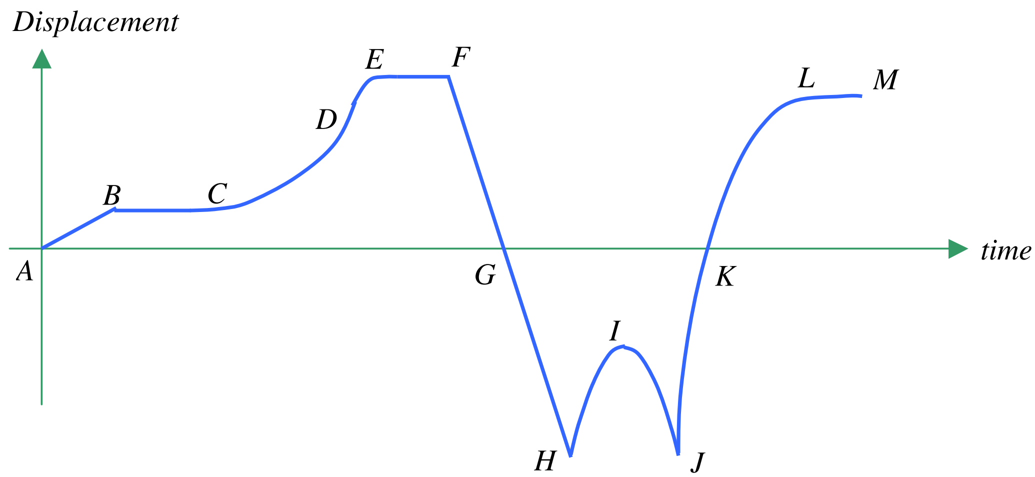


Answer:



Example

Describe the motion of a car. The following graph shows the relation of displacement and time.



Answer:

AB: constant velocity to the right

BC: the car stops (velocity = 0)

CD: acceleration to the right

DE: deceleration to the right

EF: the car stops (velocity = 0)

FH: constant velocity to the left

G: back to the starting point

HI: deceleration to the right

IJ: acceleration to the left

JK: deceleration to the right

K: back to the starting point

LM: the car stops (velocity = 0)

Example

A particle leaves the origin at $t = 0$ with an initial velocity $v_o = 3.6\hat{i}$, in m/s. It experiences a constant acceleration $\vec{a} = -1.2\hat{i} - 1.4\hat{j}$, in m/s^2. (a) At what time does the particle reach its maximum x coordinate? (b) What is the velocity of the particle at this time? (c) Where is the particle at this time?

Answer:

a) At maximum x coordinate, velocity along x coordinate equals zero, i.e. $v_x = 0$

by $v_x = v_{ox} + a_x t$

$\Rightarrow \; t = \dfrac{3.6}{1.2} = 3\text{s}$

b) At that moment, velocity of the particle equals v_y

by $v_y = v_{oy} + a_y t$

$\therefore \; v_y = 0 - 1.4 \times 3 = -4.2\text{m/s}$

c) Let the position of the particle be $\vec{S} = S_x \hat{i} + S_y \hat{j}$

by $S = ut + (1/2)at^2$

$S_x = (3.6)(3) - 0.5 \times 1.2 \times 3^2 = 5.4$

$S_y = (0)(3) - 0.5 \times 1.4 \times 3^2 = -6.3$

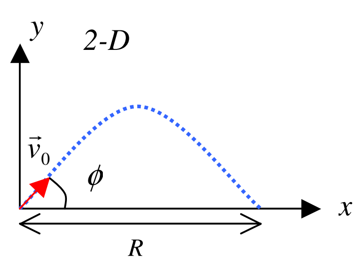
Hence, the position of the particle is $\vec{S} = 5.4\hat{i} - 6.3\hat{j}$.

1.6 Projectile

A projectile is a particle with initial velocity v_0 and its path is determined by gravity.

The acceleration $\boldsymbol{a} = \boldsymbol{g}$ (constant)

Or, we can rewrite it as $\begin{cases} a_x = 0 \\ a_y = g \end{cases}$



Now, we are going to determine the trajectory of the particle.

Useful equation: $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{a}t$

(a) x-direction

$v_x = v_{0x} + a_x t$

$v_{0x} = v_0 \cos\phi$, $a_x = 0$

$\therefore v_x = v_0 \cos\phi$ (It is a constant.)

(b) y-direction

$v_y = v_{0y} + a_y t$

$v_{0y} = v_0 \sin\phi$, $a_y = -g$ (Acceleration due to gravity)

$\therefore \quad v_y = v_0 \sin\phi - gt$

At any time t, the speed is given by

$$v = \sqrt{v_x^2 + v_y^2} = (v_0^2 - 2v_0 gt\sin\phi + g^2t^2)^{1/2}$$

(c) The trajectory

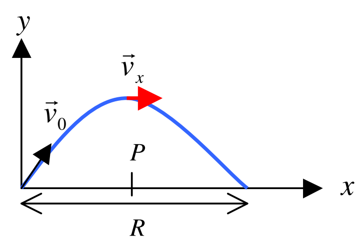
$$\because \boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \tfrac{1}{2}\boldsymbol{a}t^2$$

$$\therefore \begin{cases} x = x_0 + v_{0x}t + \tfrac{1}{2}a_x t^2 = (v_0\cos\phi)t & (A) \\ y = y_0 + v_{0y}t + \tfrac{1}{2}a_y t^2 = (v_0\sin\phi)t - \tfrac{1}{2}gt^2 & (B) \end{cases}$$

From (A), $t = \dfrac{x}{v_0\cos\phi}$, substitute it into (B)

$$\boxed{y = x\tan\phi - \frac{g}{2v_0^2\cos^2\phi}x^2}$$

Trajectory equation



- To calculate the range, we plug in $y = 0$, the solutions are

(a) $x = 0$

(b) $x = R = \dfrac{2v_0^2}{g}\sin\phi\cos\phi$

Using the identity $\sin 2\phi = 2\sin\phi\cos\phi$

$$R = \frac{v_0^2}{g}\sin 2\phi$$

- Maximum height

$$v_y = v_0\sin\phi - gt$$

At $t = 0$, $v_y = v_0 \sin\phi > 0 \uparrow$

as t increases, v_y becomes smaller, but $v_y > 0 \uparrow$

at $t_0 = \dfrac{v_0\sin\phi}{g}$, $v_y = 0$,

when $t > t_0$, $v_y < 0 \downarrow$

at $t_0 = \dfrac{v_0\sin\phi}{g}$, the projectile reaches the highest point.

$$
\begin{aligned}
y &= (v_0 \sin\phi)t - \frac{1}{2}gt^2 \\
&= \frac{v_0^2 \sin^2\phi}{g} - \frac{1}{2}g\frac{v_0^2 \sin^2\phi}{g^2} \\
&= \frac{v_0^2 \sin^2\phi}{2g}
\end{aligned}
$$

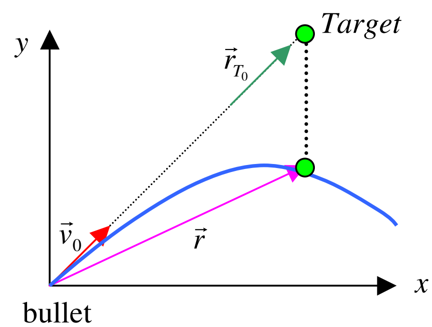
In our calculations, we have assumed that $a = g = \text{constant}$. In reality, the resistance from air "Drag force" becomes important as speed increases.

Example

Can you accurately shoot a free falling object? What is the condition?

Initial condition

$$\boldsymbol{v}_{T_0} = 0, \quad \boldsymbol{v}_{b_0} = \boldsymbol{v}_0$$
$$\boldsymbol{r}_{b_0} = 0, \quad \boldsymbol{r}_{T_0}$$



We use $\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{a}t^2$, where $\boldsymbol{a} = -g\hat{j}$

For the bullet, $\boldsymbol{r} = \boldsymbol{v}_0 t - \frac{1}{2}gt^2\hat{j}$ (1)

For the target, $\boldsymbol{r}_T = \boldsymbol{r}_{T_o} - \frac{1}{2}gt^2\hat{j}$ (2)

$(1) - (2) \Rightarrow \boldsymbol{r} - \boldsymbol{r}_T = \boldsymbol{v}_0 t - \boldsymbol{r}_{T_0}$

For the bullet and the target to occupy the same position in space or collision, we require $\boldsymbol{r} = \boldsymbol{r}_T$ at some time t. This means that $\boldsymbol{v}_0 t - \boldsymbol{r}_{T_0} = 0$

this equation gives two conditions:

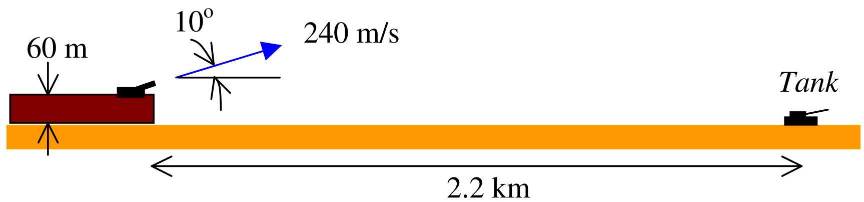
(a) $\boldsymbol{v}_0 \,//\, \boldsymbol{r}_{T_0}$

(b) $t = \dfrac{r_{T_0}}{v_0}$

As long as $\boldsymbol{r}_{T_0} \,//\, \boldsymbol{v}_0$ (aim at the target), there will be a collision!

Example

An antitank gun is located on the edge of a plateau that is 60.0 m above a surrounding plain; see figure. The gun crew sights an enemy tank stationary on the plain at a horizontal distance of 2.2 km from the gun. At the same moment, the tank crew sees the gun and starts to move directly away from it with an acceleration of 0.900 m/s^2. If the antitank gun fires a shell with a muzzle speed of 240 m/s at an elevation angle of 10.0^{o} above the horizontal, how long should the gun crew wait before firing if they are to hit the tank?



Answer:

Time for the shell to hit the ground = t

$-60 = (240 \sin 10°)\, t + 0.5\,(-9.8)\, t^2$

$t = 9.76\ \text{s}$

Horizontal distance traveled by the shell = $(240\cos 10°) \times 9.76 = 2306.78$ m

Distance traveled by the tank for it to be hit is = $2306.78 - 2200 = 106.78$ m

Let the time required for the tank to cover a distance 106.78 m is t'

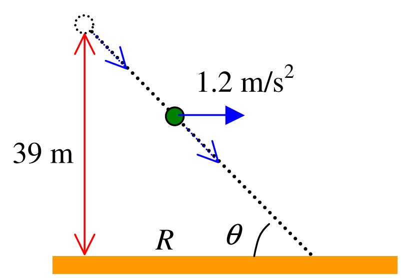
$106.78 = 0.5 \times 0.9 \times (t')^2 = 0.45(t')^2$

$t' = 15.4$ s

Hence, the time for the gun crew to wait = $t' - t = 5.64$ s

Example

A ball is dropped from a height of 39.0m. The wind is blowing horizontally and imparts a constant acceleration of 1.2m/s^2 on the ball. (a) Show that the path of the ball is a straight line and find the values of R and θ. (b) How long does it take for the ball to reach the ground? (c) With what speed does the ball hit the ground?



Answer:

a) By using the formula $S = ut + (1/2)at^2$, we can find the displacement of the ball.

Displacement along x direction S_x is

$S_x = 0.5 \times 1.2 \times t^2 = 0.6t^2$

Displacement along y direction S_y is

$S_y = 0.5 \times 9.8 \times t^2 = 4.9t^2$

$$\frac{S_y}{S_x} = \frac{4.9t^2}{0.6t^2} = 8.167 = \text{constant}$$

As slope is constant, therefore, the path is a straight line.

$$\frac{39}{R} = 8.167$$
$$R = 4.78\text{m}$$
$$\tan\theta = 8.167$$
$$\theta = 83.02^o$$

b) By $S = ut + (1/2)at^2$

$$R = 0.5 \times 1.2 \times t^2 = 0.6t^2$$
$$t = 2.8\text{s}$$

c) By $v^2 - u^2 = 2aS$

$$v_x^2 = 2 \times 1.2 \times 4.78 = 11.46$$
$$v_x = 3.39\text{m/s}$$

$$v_y^2 = 2 \times 9.8 \times 39 = 764.4$$
$$v_y = 27.65\text{m/s}$$

$$\text{speed of the ball hit the ground} = \sqrt{v_x^2 + v_y^2} = 27.85\text{m/s}$$

Example

You throw a ball with a speed of 25.3 m/s at an angle of 42.0° above the horizontal directly toward a wall as shown in figure. The wall is 21.8 m from the release point of the ball.

(a) How long is the ball in the air before it hits the wall?

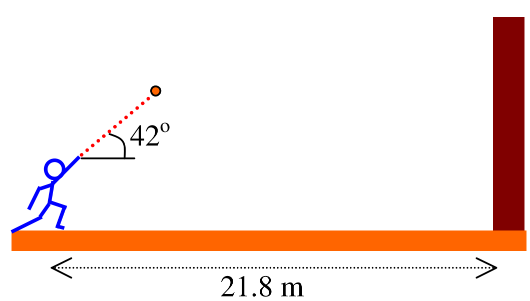
(b) How far above the release point does the ball hit the wall?

(c) What are the horizontal and vertical components of its velocity as it hits the wall?

(d) Has it passed the highest point on its trajectory when it hits?

Answer:

a)
$$\text{time taken} = \frac{\text{distance travelled}}{\text{velocity}} = \frac{21.8}{25.3\cos 42°} = 1.16\text{s}$$



b) By $S = ut + (1/2)at^2$

$$S_y = (25.3\sin 42°)(1.16) - 0.5\times 9.8\times 1.16^2$$
$$= 13.04\text{m}$$

c) horizontal velocity $= 25.3\cos 42° = 18.8\text{m/s}$

By $v = u + at$

vertical velocity $= (25.3\sin 42°) - 9.8\times 1.16$
$= 5.57\text{m/s}$ (pointing upwards)

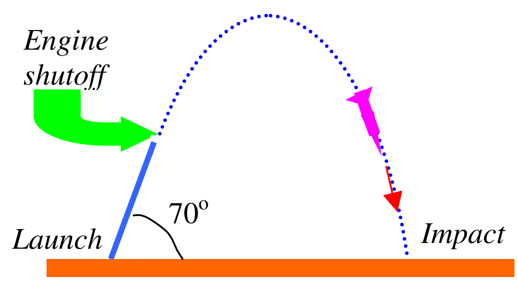
d) No change in sign of velocity in *y*-direction when the ball hits the wall, therefore the ball does not reach its maximum height in *y*-direction.

Example

A rocket is launched from rest and moves in a straight line at 70.0° above the horizontal with an acceleration of 46.0m/s^2. After 30.0s of powered flight, the engines shutoff and the rocket follows a parabolic path back to the Earth; see figure. (a) Find the time of flight from launching to impact. (b) What is the maximum altitude reached? (c) What is the distance from launch pad to impact point? (Ignore the variation of *g* with altitude.)

Answer:

a) By $S = ut + (1/2)at^2$ and $v = u + at$



Vertical displacement when the engines shutoff
$= 0.5\times 46\sin 70°\times 30^2$
$= 19451.6\text{m}$

Vertical velocity when engines shutoff
$= 46\sin 70°\times 30$
$= 1296.8\text{m/s}$

Let the time taken to impart the ground after the engines shutoff

$-19451.6 = (1296.8)t + 0.5(-9.8)t^2$
$t = 278.9\text{ s}$ or -14.2 s(rejected)

Therefore, the total time taken = 278.9 + 30 = 308.9 s

b) When reaches maximum altitude, the vertical velocity equals zero.

The vertical distance traveled for the rocket to reach maximum height after the engine shut off is:

$1296.8^2 = 2(9.8)S'$
$S' = 8.58\times 10^4$ m

Maximum altitude = 19451.6 + 85800.5 = 1.05×10^5 m

c) Horizontal distance traveled before the engines shut off is:

$$x_1 = 19451.6 / \tan 70°$$
$$= 7079.8\,\text{m}$$

Horizontal velocity of the rocket when the engine shut off is:

$$v_x = 46(\cos 70°)(30) = 472.0\ \text{m/s}$$

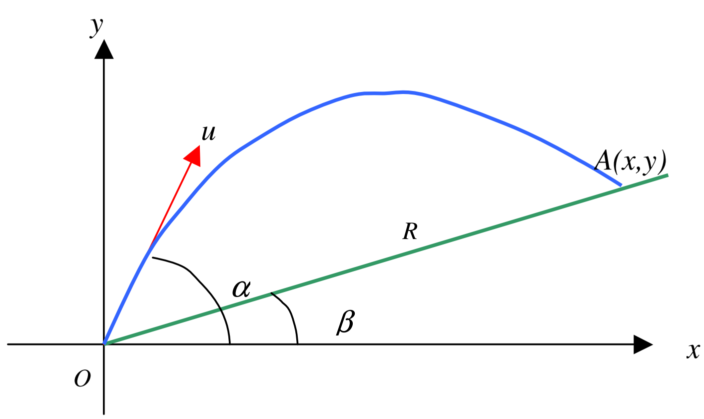
Horizontal distance traveled after the engine shut off is:

$$x_2 = 472.0 \times 278.9 = 1.32 \times 10^5\ \text{m}$$

Distance from the launch pad to the impact pad = $x_1 + x_2$ = 1.39×10^5 m

1.7 Range on an inclined plane

If a particle is projected from a point O with velocity u at an elevation α to the horizontal, we may find its range R on a plane through O inclined at an angle β to the horizontal. The vertical plane of motion contains the line of greatest slope of the plane.



Assume the particle meets the plane at a point A at time t and (x,y) is the coordinates of A, we have

$$x = u(\cos\alpha)\,t, \qquad y = u(\sin\alpha)\,t - \frac{1}{2}gt^2,$$

As $y = x\tan\beta$, hence, we have $y = u\cos\alpha(\tan\beta)\,t = u\,(\sin\alpha)t - \frac{1}{2}gt^2$.

Whence t, the time of flight is given by

$$t = \frac{2u}{g}(\sin\alpha - \cos\alpha \tan\beta) = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}.$$

Substitute t in the first equation,

$$\begin{aligned} x &= u\cos\alpha \cdot \frac{2u\sin(\alpha-\beta)}{g\cos\beta} \\ &= \frac{2u^2}{g}\cdot\frac{\cos\alpha\sin(\alpha-\beta)}{\cos\beta}. \end{aligned}$$

The range on the inclined plane, R, is given by $R=OA=x \sec\beta$

$$\begin{aligned} R &= \frac{2u^2}{g}\cdot\frac{\cos\alpha\sin(\alpha-\beta)}{\cos^2\beta} \\ &= \frac{u^2}{g}\cdot\left\{\frac{\sin(2\alpha-\beta)-\sin\beta}{\cos^2\beta}\right\}. \end{aligned}$$

For a given velocity of projection, the range is maximum when $\sin(2\alpha-\beta)=1$, i.e. $2\alpha-\beta=\frac{\pi}{2}$.

Hence the angle of projection for maximum range is $\alpha=\frac{\pi}{4}+\frac{\beta}{2}$.

For this value of α, the range is

$$\begin{aligned} R_{\max} &= \frac{u^2}{g}\cdot\left\{\frac{1-\sin\beta}{\cos^2\beta}\right\} \\ &= \frac{u^2(1-\sin\beta)}{g\,(1-\sin\beta)(1+\sin\beta)} \\ &= \frac{u^2}{g\,(1+\sin\beta)}. \end{aligned}$$