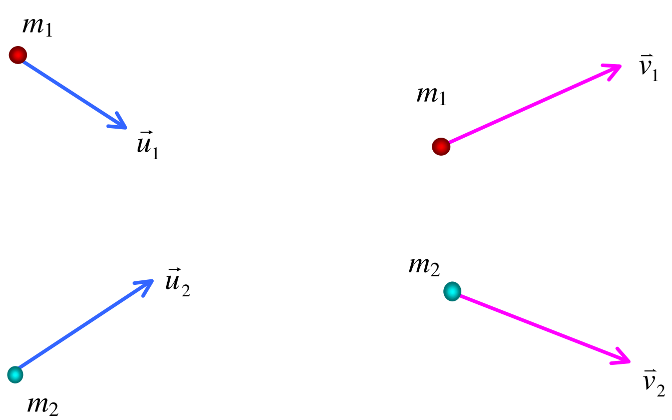
Hong Kong Physics Olympiad Lesson 3

3.1. Conservation of Momentum



Before Collision ***After Collision***

$$\Delta\vec{v}_1 = \vec{v}_1 - \vec{u}_1$$
$$\Delta\vec{v}_2 = \vec{v}_2 - \vec{u}_2$$

Consider that we are performing a collision experiment with two particles (not necessary identical particles) on a two-dimensional plane, say, smooth table. If the initial velocity vectors of the two particles were labeled as $\vec{u}_1$ and $\vec{u}_2$ respectively, then after collision, their velocity were found to be $\vec{v}_1$ and $\vec{v}_2$ respectively. After performing numerous trials with different initial velocities and final velocity being measured, it was found that:

(1) $\Delta\vec{v}_1$ is always in opposite direction of $\Delta\vec{v}_2$

(2) $\left|\dfrac{\Delta\vec{v}_1}{\Delta\vec{v}_2}\right| = \text{constant}$

We can repeat the experiment by changing different particles and we found that different particles have different degree of resistance to change its magnitude of the velocity after the collision. Or we can assign the proportionality constant such that:

$$\left|\frac{\Delta\vec{v}_1}{\Delta\vec{v}_2}\right| = \frac{m_2}{m_1}$$

where m_1 and m_2 are then called the inertia mass of the particles, which is a measure of the resistance to change the velocity magnitude during an interaction with another particle. From this experiment, we also discover a conservation law if we define a physical quantity called 'momentum' by: $\vec{p} = m\vec{v}$.

- **The mathematics behind**

During the collision, the forces act on each other are with the same magnitude but opposite in direction. This is the Newton's third law. Hence, we have

$$\frac{m_1 \Delta \vec{v}_1}{\Delta t} = -\frac{m_2 \Delta \vec{v}_2}{\Delta t}$$

$$m_1 \Delta \vec{v}_1 = -m_2 \Delta \vec{v}_2$$

Substituting Δv_1, Δv_2 and rearrange the equation, we obtain

$$m_1(\vec{v}_1 - \vec{u}_1) = -m_2(\vec{v}_2 - \vec{u}_2) \quad \text{or} \quad \sum_i \Delta \vec{p}_i = 0$$

That is,

$$m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2 \quad \text{or} \quad \sum_i \vec{p}_i = \text{constant}$$

Example

An ant lands on one end of a floating 4.75g stick. After sitting at rest for a moment, it runs toward the other end with a speed of 3.8cm/s relative to the still water. The stick moves in the opposite direction at 0.12cm/s. What is the mass of the ant?

Answer:

The total momentum of the system before the ant runs on the stick is zero. By conservation of linear momentum, the total momentum of the system after the ant runs on the stick equals zero. Hence we can write

$p_a + p_s = 0$,

$$m_a v_a + m_s v_s = 0,$$

Substituting v_a, v_s and m_a, we obtain

$$m_a(3.80)+4.75(-0.12) = 0$$

$$m_a = 0.15\text{g}$$

3.2 Collisions

Elastic collision: In an elastic collision, the momentum and the kinetic energy are conserved.
Inelastic collision: In an inelastic collision, the momentum is conserved but the kinetic energy is not conserved.

Example

Two particles, whose masses are 5 kg and 7 kg are moving on the same line with speed 30m/s and 20 m/s, respectively, when they collide. Assuming that the particles couple together after impact, find their common velocity after impact if they were

(a) moving in the same direction,

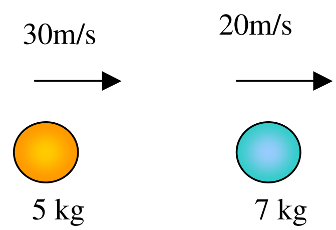
(b) moving in opposite directions.

Answer:

(a) By the conservation momentum,

$$(5 + 7)V = 5(30) + 7(20)$$

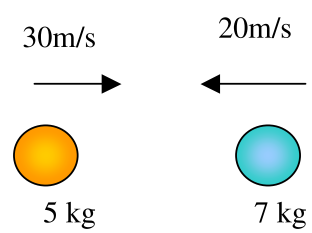
hence, $V = 145/6$ m/s



(b) By the conservation of momentum,

$$(5 + 7)V = 5(30) - 7(20)$$

hence, $V = 5/6$ m/s



Example

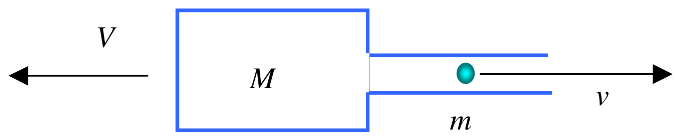
A gun of mass M fires a shell of mass m and recoils horizontally. If the shell travels with speed v relative to the barrel, find the speed with which the barrel begins to recoil if

(a) the barrel is horizontal,

(b) the barrel is inclined at an angle α to the horizontal.

Answer:

(a)



Let the barrel be recoiling with speed V. The speed of the shell as it leaves the barrel is $v - V$. Before firing the shell, the gun is at rest and the total momentum is zero. By the conservation of momentum,

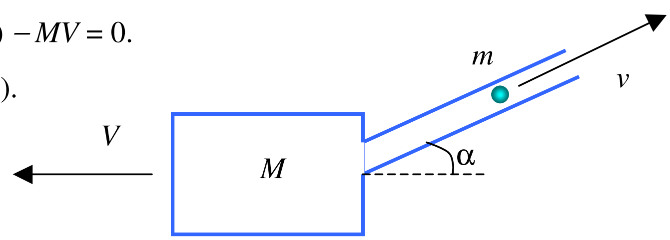
$$m(v - V) - MV = 0.$$

Hence $V = mv / (M + m)$.

(b) When the gun is inclined at angle α. The shell leaves the barrel with a velocity which is the resultant of two components, v and V. By the conservation of momentum in the direction of recoil,

$$m(v \cos\alpha - V) - MV = 0.$$

Hence $V = mv \cos\alpha \,/\, (M + m)$.



Example

A and B are two particles, of mass 4 kg and 8 kg respectively, lying in contact on a smooth horizontal table, and connected by a string 3 m long. B is 7 m from the smooth edge of the table and is connected by a taut string passing over the edge to a particle C of mass 4 kg hanging freely. If the system is released from rest, find the speed with which A begins to move.

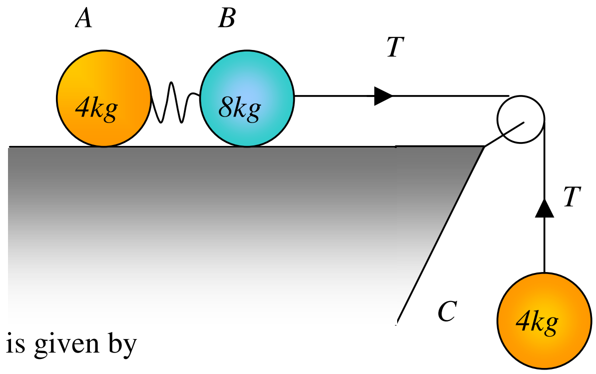
Answer:

When B is in motion and less than 3 m from A, its acceleration is given be the equations

$$4g - T = 4a,$$
$$T = 8a,$$

where T is the tension in the string, and hence

$$a = \frac{1}{3}g = 3.27\, m/s^2$$



Hence, when B has moved 3 m, its velocity is given by

$$v^2 = 2 \times \frac{9.81}{3} \times 3,$$
$$v = 4.43\, m/s,$$

and this will also be the velocity of the mass C hanging vertically.

The impulse in the string joining B to C when the string AB becomes taut will give a certain horizontal momentum to B and take away the same amount of vertical momentum from C. Hence, we may use the conservation of momentum as if all three particles were moving in the same straight line. If v m/s be their common velocity after A has been brought into motion, we have

$$(4+8+4)v = (8+4)4.43,$$
$$v = 3.32\, m/s.$$

This is therefore the speed with which A begins to move.

3.3 Impulse

Impulse is defined as the change in momentum, e.g.

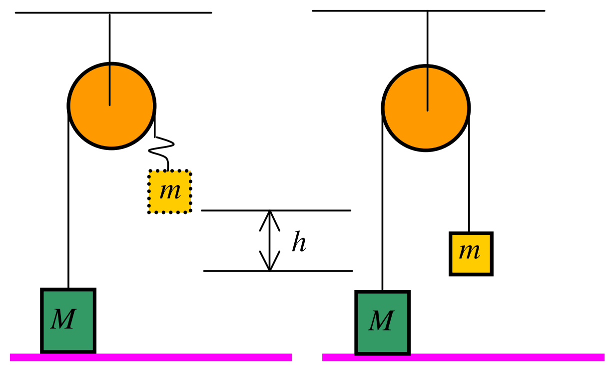
$$\boldsymbol{I} = \Delta \boldsymbol{p} = m\boldsymbol{v} - m\boldsymbol{u} = \boldsymbol{F}\Delta t$$

The unit of impulse is *Ns* or *kg m/s*.

Example

A particle of mass M lying on the ground is connected, by means of a light inextensible string passing over a smooth pulley to a mass m. After the mass m has fallen through a height h, the string tightens and the mass M begins to rise. Find the impulse applied to M when the string tightens and the initial speed.

Answer:



The velocity of mass m just before the string tightens is given by

$$v^2 = 0^2 + 2gh$$

i.e. $v = \sqrt{2gh}$.

By the conservation of momentum, we have

$$mv + M(0) = mV + MV ,$$

where V is the velocity of the system just after the string tightens.

Substituting the expression of v, we obtain $V = \dfrac{m\sqrt{2gh}}{m+M}$.

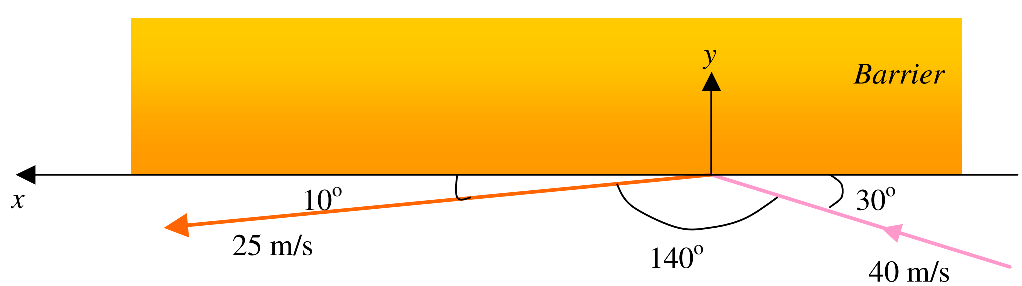
If I is the impulse, we can write $I = MV - M(0) = \dfrac{mM\sqrt{2gh}}{m+M}$.

Remarks:

The impulse on m = $mV - mv = \dfrac{m^2\sqrt{2gh}}{m+M} - m\sqrt{2gh} = -\dfrac{mM\sqrt{2gh}}{m+M}$. Of course, it is negative! As the velocity V is smaller than v.

Example

A racing car of mass 1200 kg traveling on a horizontal track at 40 m/s, strikes a vertical crash barrier at an angle of 30 degrees and rebounds with a speed of 25 m/s. If the direction of the rebound makes an angle of 140 degrees with the original direction of motion, find the impulse given to the car by the barrier.



Answer:

The velocity of car before crash:

$$\boldsymbol{v_o} = 40[\hat{i}\cos 30 + \hat{j}\sin 30]$$
$$= 40(\hat{i}\sqrt{3}/2 + \hat{j}/2)$$
$$= (34.64\hat{i} + 20\hat{j})\,m/s$$

The velocity of car after crash:

$$\boldsymbol{v_1} = 25[\hat{i}\cos 10 - \hat{j}\sin 10]$$
$$= 25(0.9848\hat{i} - 0.1736\hat{j})$$
$$= (24.62\hat{i} - 4.34\hat{j})\,m/s$$

Now $\boldsymbol{I} = m(\boldsymbol{v_1} - \boldsymbol{v_o})$

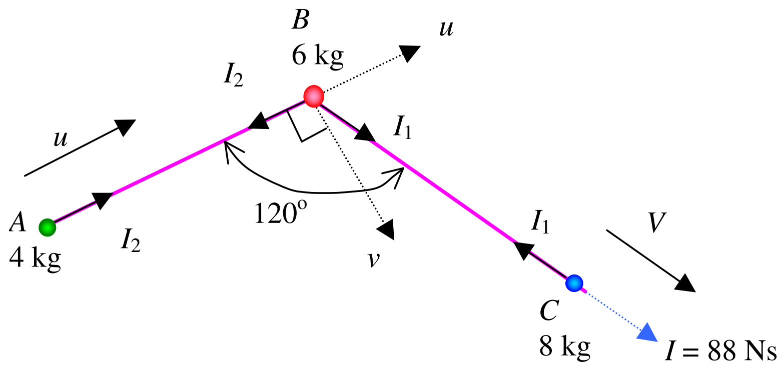
Hence impulse received by car

$$= 1200[(24.62 - 34.64)\,\hat{i} + (-4.34 - 20\,\hat{j})]$$
$$= 1200[-10.02\hat{i} - 24.34\hat{j}]$$
$$= (-12024\hat{i} - 29208\hat{j})\,Ns$$

Example

Three particles *A*, *B* and *C*, of masses 4, 6 and 8 kg, respectively, lie at rest on a smooth horizontal table. They are connected by taut light inextensible strings *AB* and *BC* and $\angle ABC = 120^o$. An impulse $\boldsymbol{I}$ is applied to *C*. If the magnitude of $\boldsymbol{I}$ is 88 Ns and it acts in the direction *BC*, find the initial speeds of *A*, *B* and *C*.

Answer:



Let $\boldsymbol{I_1}$ and $\boldsymbol{I_2}$ be the impulsive tensions in the strings *BC* and *AB* respectively. Since *A* is acted on only by $\boldsymbol{I_2}$, its initial speed u will be in the direction of *AB*. Since the string *AB* is taut, *B* must have a speed u in the direction *AB*. Also, let *B* have a speed v in a direction perpendicular to *AB* [this is necessary because $\boldsymbol{I_1}$ acts in a different direction to $\boldsymbol{I_2}$]. Finally, since both $\boldsymbol{I}$ and $\boldsymbol{I_1}$, the impulses acting on *C*, have the same direction *BC*, let *C* have an initial speed *V* in the direction *BC*.

Since *BC* is taut, the speeds of *B* and *C* in the direction *BC* are equal.

$\therefore \quad u\cos 60 + v\cos 30 = V$

$\therefore \quad u/2 + v\sqrt{3}/2 = V. \qquad (i)$

Considering the motion of *A*,

$4u = I_2. \qquad$ (ii)

Considering the motion of *B* along and perpendicular to *AB*,

$6u = I_1 \cos 60 - I_2.$

$\therefore \quad 6u = I_1/2 - I_2 \qquad (iii)$

$6v = I_1 \cos 30$

$\therefore \quad 6v = I_1\sqrt{3}/2. \qquad (iv)$

Considering the motion of *C*,

$8V = 88 - I_1. \qquad$ (v)

Eliminating I_2 from equations (ii) and (iii)

$6u = I_1/2 - 4u$

$\therefore \quad I_1 = 20u. \qquad (a)$

Substituting from equation (a) in equation (iv),

$6v = 20u\sqrt{3}/2$

$\therefore \quad v = \dfrac{5}{\sqrt{3}}u \qquad (b)$

Substituting from equation (b) in equation (i),

$$\frac{u}{2} + \frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} u = V$$

$V = 3u \qquad (c)$

Substituting for *V* and I_1 [equations (a) and (c)] in equation (v),

$8 \cdot 3u = 88 - 20u$

$\therefore \quad 44u = 88$

$\therefore \quad u = 2.$

From (b) $\quad v = 10/\sqrt{3}$.

From (c) $V = 6$.

$\therefore$ Speed of A is 2m/s,

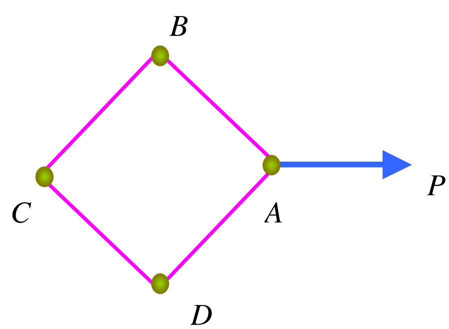
Speed of B is $\sqrt{u^2 + v^2} = \sqrt{4 + \dfrac{100}{3}}$

$$= 4\sqrt{\frac{7}{3}} m/s.$$

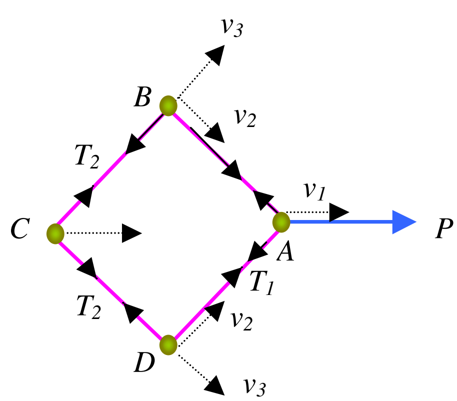
Speed of C is 6m/s.

Example

Four particles each of mass m are connected by inextensible strings. They are in the form of square as shown in the figure. If the particle A is acted by an impulse P, find the velocities of A, B, C, and D after immediately after the impulse.



Answer:



Let v_1 be the velocity of A, v_2 and v_3 be the components of the velocities of B and D, v_4 be the velocity of C after the impulse P is applied. Let T_1 and T_2 be the impulsive tensions.

Impulse consideration:

For particle A: $P - 2T_1 \cos 45 = mv_1$

$$\Rightarrow \quad P - \sqrt{2}T_1 = mv_1 \tag{1}$$

For particle B: $T_1 = mv_2$ (2)

and $\quad T_2 = -mv_3$ (3)

For particle C: $2T_2 \cos 45 = mv_4$

and $\quad \sqrt{2}T_2 = mv_4$ (4)

Velocity consideration:

Along BA: $\quad v_1 \cos 45 = v_2$

$$\Rightarrow \quad v_1 = \sqrt{2}v_2 \tag{5}$$

Along CB: $\quad v_3 = v_4 \cos 45$

$$\Rightarrow \quad \sqrt{2}v_3 = v_4 \tag{6}$$

Substitute (2) into (1) and substitute (3) into (4):

$$P - \sqrt{2}mv_2 = mv_1 \tag{7}$$

and $\quad -\sqrt{2}mv_3 = mv_4$

$$-\sqrt{2}v_3 = v_4 \tag{8}$$

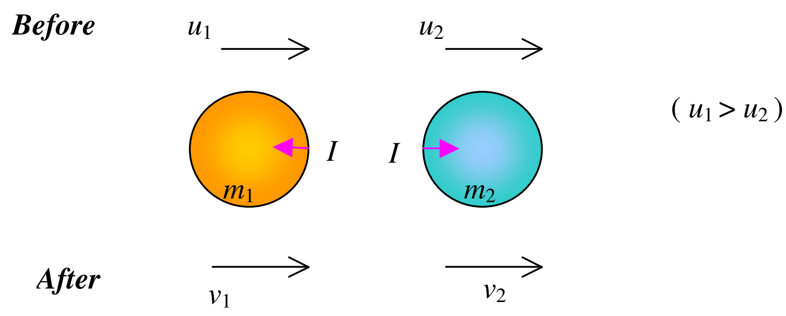
After solving (5), (6), (7) and (8) for the velocities, we get

$$v_1 = \frac{P}{2m}, \qquad v_2 = \frac{P}{2\sqrt{2}m}, \qquad v_3 = v_4 = 0.$$

The initial velocity of particle A is $v_1 = \dfrac{P}{2m}$. The initial velocities of B and D are $v_2 = \dfrac{P}{2\sqrt{2}m}$.

The velocity of particle C is zero.

3.4 Coefficient of restitution (e)



The above figure depicts two bodies. Since the total momentum is conserved, we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2.$$

This one equation is not sufficient to calculate v_1 and v_2 and we have recourse to Newton's experimental law. If the velocities both before and after impact are taken relative to the same body, then, for two bodies impinging directly, their relative velocity after impact is equal to a constant (e) times their relative velocity before impact and in the opposite direction. e is known as the coefficient of restitution.

$$v_1 - v_2 = -e\,(u_1 - u_2)$$

In the case of oblique impact, the result holds for the components the velocities in the direction of the common normal at impact. The value of e has to be found by experiment and varies from 0 for completely inelastic bodies to practically 1 for nearly perfectly elastic bodies. Note that the quantities u_1, u_2, v_1 and v_2 mentioned above are in the same direction.

Example

A smooth sphere strikes an identical sphere initially at rest. If the velocity of the moving sphere before the impact is 2 m/s at 45° to the line of center AB, and e = 0.6, find the velocities of the spheres after the impact.

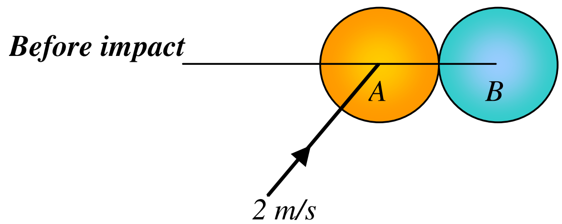
Answer:

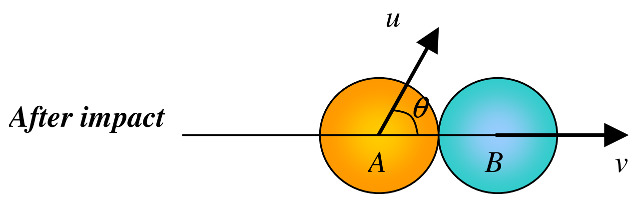
The stationary sphere receives an impulse in the direction of AB. So this sphere moves in the direction of *AB, e.g.* $\boldsymbol{V}$, after impact.

Consider the momentum of the first sphere perpendicular to AB,

$$m(2\sin 45) = mu\sin\theta$$

$$\Rightarrow \qquad u\sin\theta = \sqrt{2} \qquad (m\text{: mass of the sphere})$$





From the conservation of momentum along AB,

$$m(2\cos 45) = mu\cos\theta + mv$$

and from Newton's law of restitution,

$$0.6(2\cos 45) = v - u\cos\theta$$

$$v + u\cos\theta = 2\cos 45$$

and $\qquad v - u\cos\theta = 1.2\cos 45$

$\Rightarrow \qquad v = 1.6\cos 45$ and $u\cos\theta = 0.4\cos 45$

i.e. $\qquad v = 1.13$ and $u\cos\theta = 0.283$

Since $\qquad u\sin\theta = \sqrt{2}$, $\tan\theta = 5$

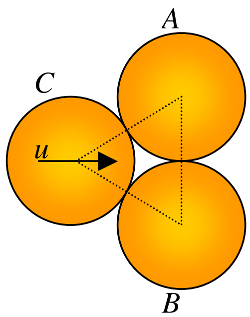
$\theta \approx 79^o$ and $u = 1.44$

Thus the velocity of the second sphere is 1.13 m/s along AB, and that of the first sphere is 1.44 m/s at 79° to AB.

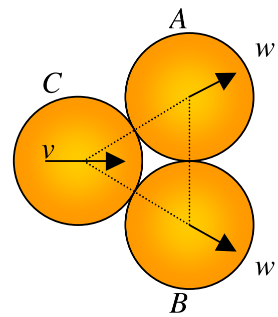
Example

Three identical spheres are arranged as shown in figure. If sphere C is projected with velocity u while A and B are at rest. Given that the coefficient of restitution is e for each sphere, find the subsequent velocities of each sphere. Show also that the condition for sphere C to pass through and beyond the two spheres A and B is $e < 1/9$.

Answer:



Before impact



After impact

Let u be the velocity of sphere C before impact, v the velocity of sphere C after impact and w the velocities of A and B after impact.

By conservation of momentum:

$$mu = mv + 2mw\cos 30 \qquad \Rightarrow v + 2w\cos 30 = u \qquad (1)$$

By Newton's law of restitution:

$$e = -\frac{w - v\cos 30}{0 - u\cos 30} \Rightarrow w - v\cos 30 = eu\cos 30 \qquad (2)$$

Solving (1) and (2), we get $v = \frac{u}{5}(2-3e)$ and $w = \frac{\sqrt{3}u}{5}(1+e)$

Thus, the velocity of C after impact is $\frac{u}{5}(2-3e)$ and the velocities of A and B after impact are the same as $\frac{\sqrt{3}u}{5}(1+e)$.

If the sphere C passes through and beyond the two spheres A and B, then

$$v > w\cos 30 \quad \Rightarrow \quad \frac{u}{5}(2-3e) > \frac{\sqrt{3}u}{5}(1+e)\cdot\frac{\sqrt{3}}{2}$$

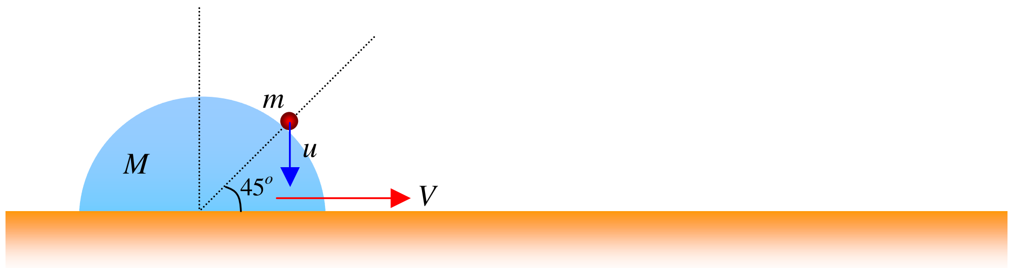
$$4 - 6e > 3 + 3e \ \Rightarrow e < \frac{1}{9}.$$

Example

A smooth uniform hemisphere of mass M is sliding with velocity V on an inelastic smooth horizontal plane with which its base is in contact. A sphere of mass m is dropped vertically from height so as to strike the hemisphere (see the figure). If the velocity of the sphere just before impact is u, and the hemisphere is stopped deadly after impact. Show that

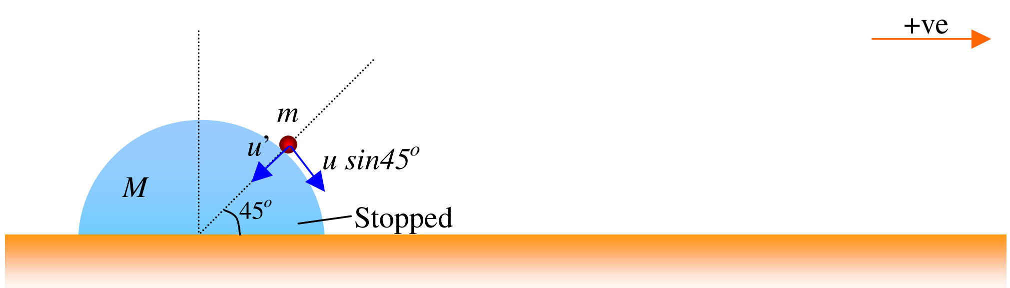
$$u = \frac{V(2M - em)}{(1+e)m},$$

where e is the coefficient of restitution.



Just before impact

Answer:



Just after impact

Consider the conservation of momentum horizontally, we have

$$MV = mu\sin 45\cos 45 - mu'\cos 45$$

$$MV = mu\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) - mu'\left(\frac{1}{\sqrt{2}}\right).$$

Hence, we obtain

$$MV = \frac{mu}{2} - \frac{mu'}{\sqrt{2}}. \qquad (1)$$

Along the line of collision, the velocities before and after collision relate the coefficient of restitution by

$$-u' - 0 = -e(-u\cos 45 - V\cos 45)$$

$$u' = e\left(-\frac{u}{\sqrt{2}} - \frac{V}{\sqrt{2}}\right)$$

Hence, we obtain

$$u' = -\frac{e}{\sqrt{2}}(u+V) \qquad (2)$$

From (1) and (2), we obtain

$$MV = \frac{mu}{2} - \frac{m}{\sqrt{2}}\left(-\frac{e}{\sqrt{2}}(u+V)\right)$$

$$= \frac{mu}{2} + \frac{me}{2}(u+V)$$

$$= \frac{m}{2}(1+e)u + \frac{m}{2}eV.$$

Rearrange the above expression, the velocity of the sphere u is given by

$$u = \frac{V(2M - em)}{(1+e)m}.$$