Hong Kong Physics Olympiad Lesson 4

4.1 Work and kinetic energy

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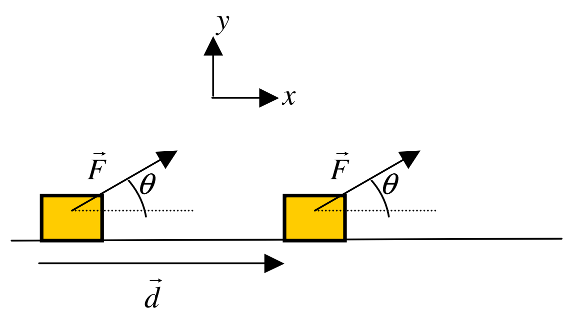
4.1 Work and kinetic energy

(1) Work: unit in joule (J)



Work done by the force ***F***

$$W = F_x d = (F\cos\theta)d = \boldsymbol{F}\cdot\boldsymbol{d}$$

F_x : the force along the displacement

Remark:

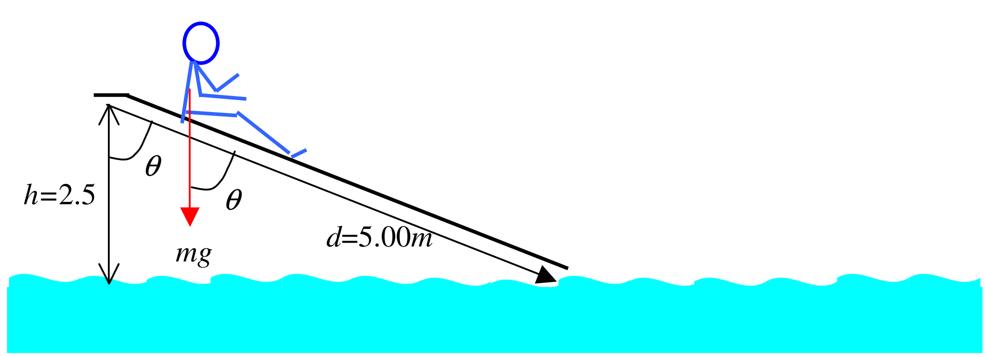
Positive work is done on an object when the point of application of the force moves in the direction of the force.

Mathematical statement: Work done is the dot product of the force and the displacement.

Dot product: $\boldsymbol{A}\cdot\boldsymbol{B} = AB\cos\theta = A_xB_x + A_yB_y + A_zB_z$

Example

A 75.0-kg person slides a distance of 5.00m on a straight water slide, dropping through a vertical height of 2.50 m. How much work does gravity do on the person?



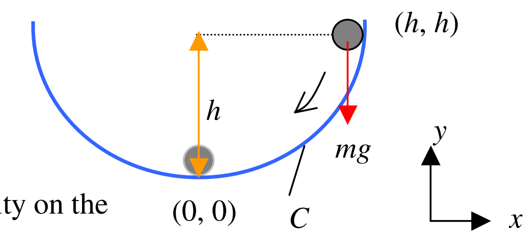
Answer:

The gravity along the displacement has a magnitude $mg \cos \theta$.

By the definition of work done, the work done of the gravity on the person is given by

$$W = (mg \cos \theta)\, d = mg\, (h / d)\, d = mgh = (75.0)(9.8)(2.50) = 1837.5 \text{ J}$$

Remark:



If $\boldsymbol{F}$ or $\boldsymbol{d}$ is not a constant, then

$$\Delta w_i = \boldsymbol{F}_i \cdot \Delta \boldsymbol{s}_i \qquad W = \sum_i \boldsymbol{F}_i \cdot \Delta \boldsymbol{s}_i$$

where $\Delta \boldsymbol{s}_i$ is extremely small. The work done of the gravity on the ball's falling down is given by

$$W = \int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_C \text{-}F\hat{\boldsymbol{j}} \cdot (\mathrm{d}x\,\hat{\boldsymbol{i}} + \mathrm{d}y\,\hat{\boldsymbol{j}}) = -F\int_h^0 \mathrm{d}y = -mg\int_h^0 \mathrm{d}y = mgh$$

(2) Power: work done per unit time

$$\text{Power} = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt} = \frac{\boldsymbol{F} \cdot d\boldsymbol{s}}{dt}$$

Since $\boldsymbol{v} = \dfrac{d\boldsymbol{s}}{dt}$, we have the power equals to the dot product of force and velocity. e.g.

$$\text{Power} = \boldsymbol{F} \cdot \boldsymbol{v}$$

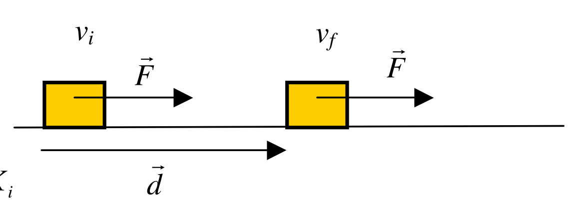
If the force and the velocity are in the same direction, $P = Fv$.

(3) Kinetic energy $\quad K \equiv \frac{1}{2} m v^2$

4.2 Work - energy theorem

If an object is moving with an acceleration a and a distance d is moved, then according to the formula $v_f^2 = v_i^2 + 2ad$, we have

$$v_f^2 = v_i^2 + 2(\frac{F}{m})d\ .$$



The work done on the object W is given by

$$W = Fd = \frac{m}{2}(v_f^2 - v_i^2) = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = K_f - K_i$$

Or, we can rewrite it as $\boxed{W = K_f - K_i = \Delta K}$

The work-energy theorem states that the total work done on the object by the forces equals the change in kinetic energy.

Proof by integration:

Suppose a particle moves in a straight line (1-D)

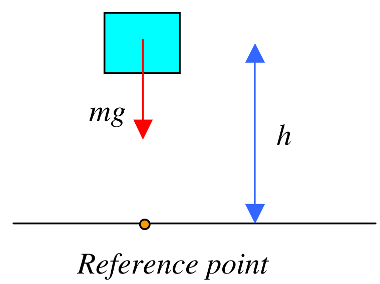
$$W = \int_{x_i}^{x_f} F(x)dx$$

$$\because F = ma = m\frac{dv}{dt}, \qquad v = \frac{dx}{dt}$$

$$\therefore W = \int_{x_i}^{x_f} m(\frac{dv}{dt})dx = \int_{x_i}^{x_f} mdv\frac{dx}{dt} = \int_{v_i}^{v_f} mvdv$$

$$= \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 = K_f - K_i$$

4.3 Potential energy



The gravitational potential energy

$$U = mgh$$

The difference of the potential energy: $\Delta U = U_f - U_i = -W$,

where W is the work done by gravity if the block is released.

In this case, W is positive.

When a spring is compressed by an amount x,

the work done of the applied force $= \int_0^x Fdx = \int_0^x kxdx = \dfrac{1}{2}kx^2$ (F: applied force)

$$U = \frac{1}{2}kx^2 \qquad \text{potential energy of a spring.}$$

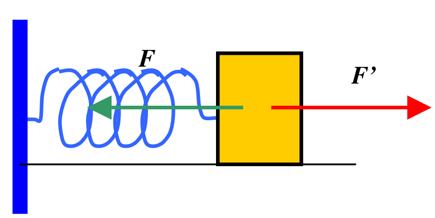
The elastic potential energy

Hooke's law states that the restoring force is proportional to the elongation of the spring from its natural length, e.g.

- $\boldsymbol{F} = -k\boldsymbol{x}$

where $\boldsymbol{F}$ is the restoring force and k is called spring constant.

If the elongation goes from zero to x, the work done by the applied force F'



$$W = \int_0^x F' dx = \int_0^x (-F) dx = \int_0^x kx dx = \tfrac{1}{2} kx^2 \Big|_0^x = \tfrac{1}{2} kx^2$$

Suppose the spring is stretched a distance x_1 initially. Then the work we have to stretch it to a greater elongation x_2 is

$$W = \int_{x_1}^{x_2} F dx = \int_{x_1}^{x_2} kx dx = \tfrac{1}{2} k{x_2}^2 - \tfrac{1}{2} k{x_1}^2$$

4.4 Total energy

$$E = K + U$$

E: mechanical energy, K: kinetic energy, U: potential energy

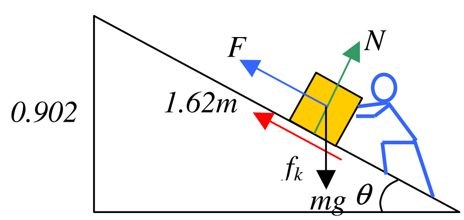
For an isolated system (no frictional force), $E = const.$ or it is conserved.

$$K + U = const.$$

This is the so-called conservation of energy.

Example

A 47.2-kg block of ice slides down an incline 1.62 m long and 0.902 m high. A worker pushes up on the ice parallel to the incline so that it slides down at constant speed. The coefficient of kinetic friction between the ice and the incline is 0.110. Find (a) the force exerted by the worker, (b) the work done by the worker on the block of ice, and (c) the work done by gravity on the ice.



Answer:

a) Angle of inclination = θ

$$\sin\theta = \frac{0.902}{1.62}$$

$$\theta = 33.8°$$

The normal reaction, $N = 47.2 \times 9.8 \times \cos 33.8° = 384.2$ N

Frictional force, $f_k = 0.110 \times 384.2 = 42.3$ N

The forces which acts on the ice:

Force exerted by the worker + frictional force = down plane force = $mg\sin\theta$

Force exerted by the worker = $47.2 \times 9.8 \sin 33.8^o - 42.3$

$= 215.02$ N

b) Work done by the worker $= -\,215.02 \times 1.62$

$= -\,348.33$ J

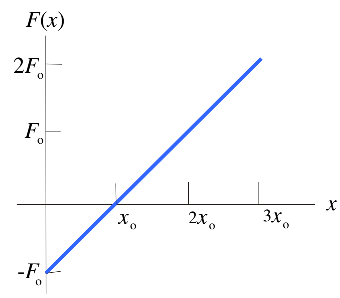
c) Work done by the gravity $= \ 47.2 \times 9.8 \times 0.902 = 417.2$ J

Example

The force exerted on an object is $F = F_o(x/x_o - 1)$. Find the work done in moving the object from $x = 0$ to $x = 3x_o$ (a) by plotting $F(x)$ and using the area under the curve, and (b) by evaluating the integral analytically.

Answer:

a) The graph shows F as a function of x if x_o is positive. The work is negative as object moves from $x = 0$ to $x = x_o$ and positive as it moves from $x = x_o$ to $x = 3x_o$. Since the area of a triangle is $(1/2)\cdot(\text{base})\cdot(\text{altitude})$, the work done for the movement from $x = 0$ to $x = x_o$ is $(-1/2)(x_o)(F_o)$ and the work done for the movement from $x = x_o$ to $x = 3x_o$ is $(1/2)(3x_o - x_o)(2F_o) = 2x_o \times F_0$.



The total work done is the sum of them, which is $(3/2)(x_o)(F_o)$.

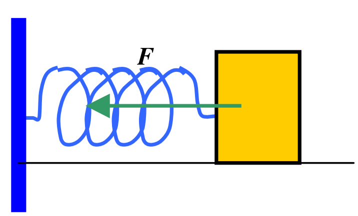
b) The integral for the work is

$$W = \int_0^{3x_o} F_o(x/x_o - 1)dx = F_o\left(x^2/2x_o - x\right)\Big|_0^{3x_o} = (3/2)(x_o)(F_o)$$

4.5 Energy diagrams

A spring is fixed at one end and the other end is attached to mass m which moves back and forth with an amplitude A

- Elastic potential energy $\qquad U(x) = \frac{1}{2}kx^2$

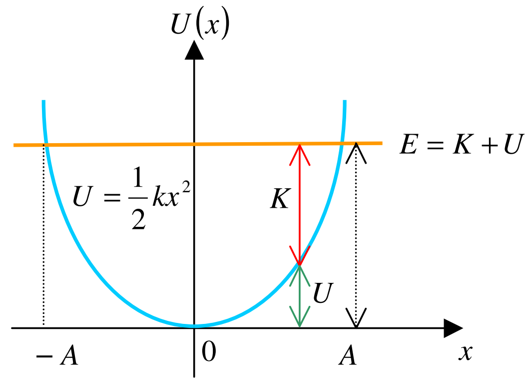


If frictional force is neglected, we have the energy of the system being conserved, e.g. E = *constant*.

When $x=0$, $U = 0$, $K = E$ and $K = \frac{1}{2}mv^2$,

v is the greatest velocity in the whole process of oscillation.

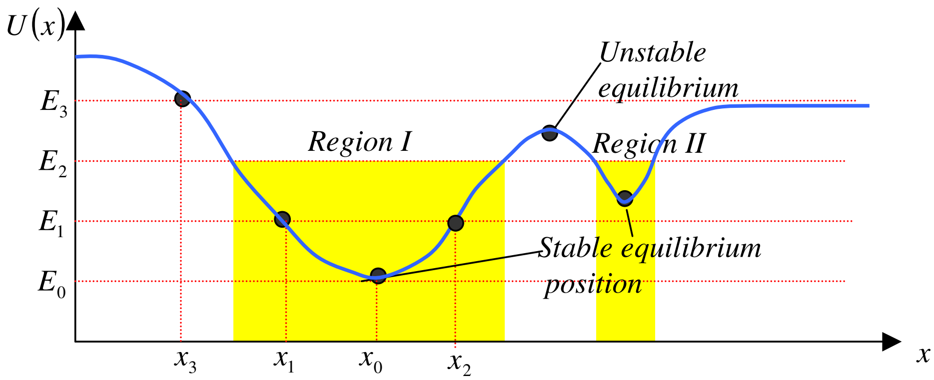
At $x = -A$, $E = max(U)$, $K = 0 \Rightarrow \quad v = 0$



- $F = -\dfrac{dU}{dx}$ = The negative slope of the curve $U(x)$ against x,
- At the origin $\dfrac{dU}{dx} = 0$, it is the equilibrium position. The resultant force is zero.
- When $x > 0$, $\dfrac{dU}{dx} > 0$, hence $F < 0$ represents that the restoring force is pointing to the left.
- When $x < 0$, $\dfrac{dU}{dx} < 0$, hence $F > 0$ represents that the restoring force is pointing to the right.
- When $x = 0$, it is called equilibrium position.

4.6 A general study of the potential energy curve

The following curve shows a profile of an arbitrary potential energy with the position.



Conservation of mechanical energy

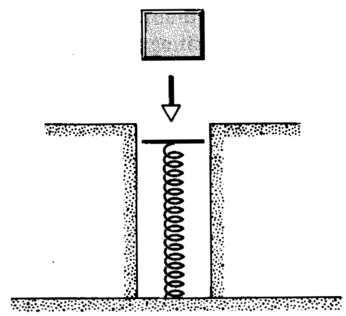
$$E = K + U(x) = const.$$

$$K = \tfrac{1}{2}mv^2 \geq 0$$

- If $E = E_0$ initially, then the particle will stay at point x_0, otherwise K has to be negative.

- When $E = E_1$, the motion of the particle is confined between $x_1 \leq x \leq x_2$, x_1 and x_2 are called turning points.
- $E = E_2$, depending on the initial condition of x, the particle will either move in region I or region II.
- $E = E_3$, the particle will move between $x = x_3$ and ∞.
- Unstable equilibrium position: Any maximum in a potential energy curve is an unstable equilibrium position.

Example



A 263-g block is dropped onto a vertical spring with force constant k = 2.52 N/cm. The block sticks to the spring, and the spring compresses 11.8 cm before coming momentarily to rest. While the spring is being compressed, how much work is done (a) by the force of gravity and (b) by the spring? (c) What was the speed of the block just before it hit the spring? (d) If this initial speed of the block is doubled, what is the maximum compression of the spring? Ignore friction.

Answer:

a) Let the compression of the spring be x. The work done by the block's weight is

$$W_1 = mgx = \left(263\times10^{-3}\right)(9.8)\left(11.8\times10^{-2}\right) = 0.304\text{ J}.$$

b) The work done by the spring is

$$W_2 = -\frac{1}{2}kx^2 = -\frac{1}{2}\left(2.52\times10^2\right)\left(11.8\times10^{-2}\right)^2 = -1.74\text{ J}.$$

c) The speed v_i of the block just before it hits the spring is given by

$$\Delta K = 0 - \frac{1}{2}mv_i^2 = W_1 + W_2$$

which yields

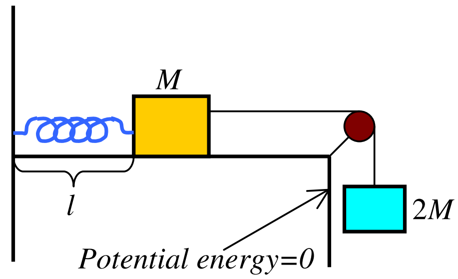
$$v_i = \sqrt{\frac{(-2)(W_1 + W_2)}{m}} = \sqrt{\frac{(-2)(0.304 - 1.74)}{263\times10^{-3}}} = 3.3\text{ m/s}$$

d) Let the new compression be $x' = cx$, where $x = 11.8\text{ cm}$, then

$$-\frac{1}{2}m(2v_i)^2 = 4(W_1 + W_2) = mgcx - \frac{1}{2}kc^2x^2 = cW_1 + c^2W_2,$$

i.e. $4(0.304 - 1.74) = 0.304c - 1.74c^2$, which gives $c = 1.91$. Thus the new compression is $x' = cx = 1.91 \times 11.8 = 22.5\,\text{cm}$.

Example



Given that M = 20 *kg*, the force constant k = 10 *kN/m*, and g = 10 m/s^2. If the frictional force is 80 *N*, and the system is held at rest with $l = l_0 + 10$ *cm*, and then released. l_0 is the unstretched length of the spring.

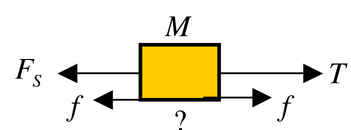
(a) What is the kinetic energy of the block, when it has moved 2.0 *cm* from its point of release?

(b) What is the kinetic energy of the block when it first slides back through the point where the spring is unstretched?

(c) What is the maximum kinetic energy attained by the block while it is sliding from its point of release to the point where the spring is unstretched?

Answer:

(a)



i) Before the release, $K_i = 0$, $x = 10cm = 0.1m$, $U_i = \frac{1}{2}kx^2 =$ 50 *Nm*, and the potential (gravitational) energy is assumed to be zero,

ii) Where does it go? $F_s = kx = 10000\frac{N}{m} \times 0.1m = 1000N$

$$T = 2Mg = 2 \times 20kg \times 10m/s^2 = 400N$$

$F_s > T$ $\quad\quad \therefore F_s - T - f = 1000 - 400 - 80 = 520N$

Final elastic potential energy $= \frac{1}{2}kx^2 = \frac{1}{2} \times 10000 \times 0.08^2 = 32N \cdot m$

$$W_f = -fd = -80N \cdot 0.02m = -1.6N \cdot m$$

Final potential energy: $\quad\quad m = 2M$, $g = 10m/s^2$

$$\begin{aligned} P.E. &= mg(0.02m) \\ &= 40kg \times 10m/s^2 \cdot 0.02m \\ &= 8N \cdot m \end{aligned}$$

Final kinetic energy: $K = \frac{1}{2}(M + 2M)v^2$

By the work-energy theorem $\Delta E = \Delta K + \Delta U = W_f$

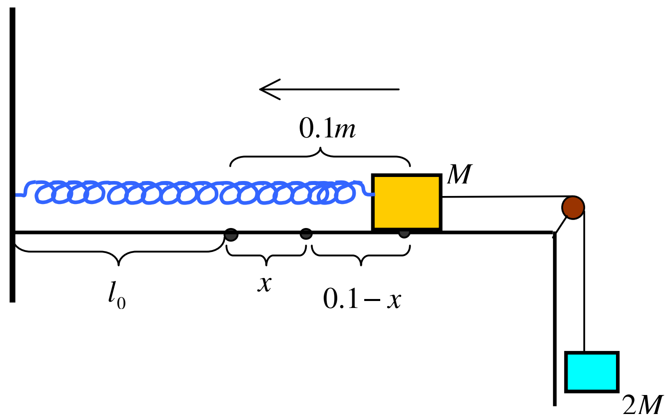
$$\tfrac{3}{2} Mv^2 + U_f - U_i = W_f \qquad (U_i = \tfrac{1}{2} \times 10000 \times 0.1^2\, N \cdot m = 50N \cdot m)$$

$$\therefore (30kg)\, v^2 + (32N \cdot m + 8N \cdot m) - 50N \cdot m = -1.6N \cdot m$$

$$\therefore (30kg)\, v^2 = 8.4N \cdot m$$

Kinetic energy of the sliding block

$$\tfrac{1}{2} Mv^2 = \frac{8.4}{3} N \cdot m$$



(b) $l = l_0$, the elastic potential energy = 0

Potential energy $= 2Mg \times (0.1m) = 40N \cdot m$

Therefore $U_f = (40 + 0)\, Nm$.

$$\Delta E = \Delta K + \Delta U = W_f$$

$$W_f = -fd = -80N \times 0.1m = -8N \cdot m$$

$$\tfrac{1}{2} \cdot 3 \cdot (20kg) v^2 + 40N \cdot m - 50N \cdot m = -8N \cdot m$$

$$\therefore (30kg) v^2 = 2N \cdot m \qquad \text{or} \qquad \tfrac{1}{2} Mv^2 = \frac{2}{3} N \cdot m$$

(c) $K_f + U_f - K_i - U_i = W_f$

We want to find $K_{f\,\max}$, assume that when elongating is x, $K_{f\,max}$ is maximum.

* $\quad U_i = \frac{1}{2} k (0.1m)^2,\ K_i = 0$

* $\quad K_f = \frac{1}{2} \times 3Mv^2$

$$U_f = 2Mg(0.1 - x) + \tfrac{1}{2} kx^2$$

$$W_f = -f(0.1 - x)$$

$$\therefore \tfrac{3}{2} Mv^2 + 2Mg(0.1 - x) + \tfrac{1}{2} kx^2 - \tfrac{1}{2} k(0.1)^2 = -f(0.1 - x)$$

or $\quad \frac{3}{2} Mv^2 = -2Mg(0.1 - x) - \frac{1}{2} kx^2 + \frac{1}{2} k(0.1)^2 - f(0.1 - x)$

N.B. $\quad y = F(x)$, $\quad \frac{dy}{dx} = 0 \Rightarrow$ Turning point, and then we can find $y_{max.}$

$$\frac{dv^2}{dx} = 0 \Rightarrow \quad -kx + 2Mg + f = 0 \quad \text{or } x = \frac{2Mg + f}{k}$$

$$\therefore \tfrac{3}{2} M v_{\max}{}^2 = \tfrac{1}{2} k \times (0.1)^2 - \tfrac{1}{2} k (\frac{2Mg + f}{k})^2 - (2Mg + f)(0.1 - \frac{2Mg + f}{k})$$

Example

A particle is set moving with kinetic energy K straight up a rough inclined plane, of inclination α and coefficient of friction μ. Prove that the work done against friction before the particle first comes to rest is

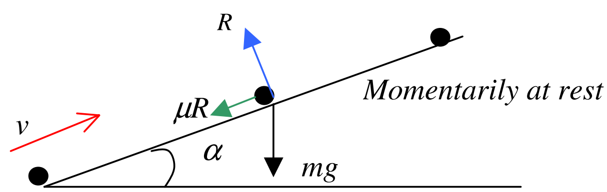
$$\frac{K\mu\cos\alpha}{\sin\alpha + \mu\cos\alpha}.$$

Answer:

Let the mass of the particle be m, the normal reaction on the plane be R, the initial velocity be v, and the distance travelled up the plane before it first comes to rest be s.

By energy conservation,

$$K = \frac{1}{2} m v^2 = mgs \sin\alpha + \mu R s,$$



where $R = mg \cos\alpha$.

Hence $K = mgs(\sin\alpha + \mu\cos\alpha)$.

The work done against friction is $\mu R s = \mu mgs \cos\alpha = \dfrac{K\mu\cos\alpha}{\sin\alpha + \mu\cos\alpha}$.

Example

Ballistic pendulum is a device which measures the speed of bullet. If the bullet sticks with the block immediately after impact, find the initial velocity of the bullet v. Given that the combined system raises a height h.

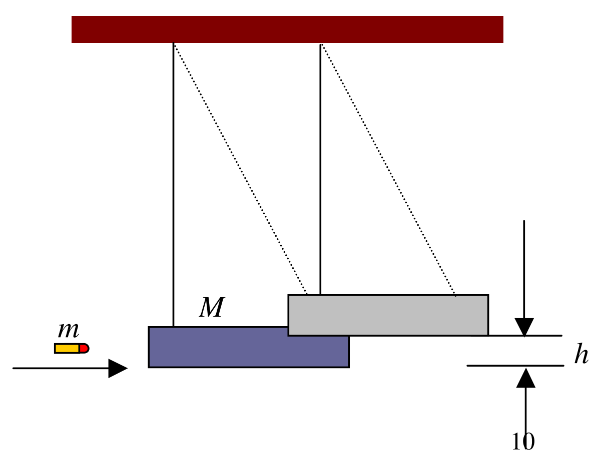
Answer:

M: A large block of wood

m: Bullet

h: The height of the block and the bullet rise

Note that m, M and h are known variables.



- This is a typical inelastic process, during the collision, the momentum is conserved.

  $mv = (M+m)V$

  V: the speed after collision

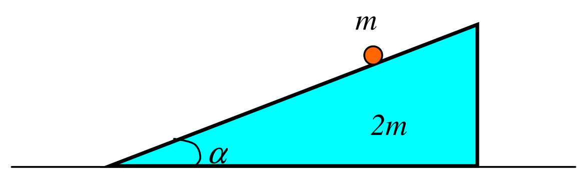
  $v = \frac{M+m}{m}V$
- After collision, the wood along with the bullet swing upward to the height h, the kinetic energy has been converted to the potential energy.
- The energy is conserved throughout the process, hence

  $\frac{1}{2}(M+m)V^2 = (M+m)gh \Rightarrow V = \sqrt{2gh}$

  $\therefore v = \frac{M+m}{m}\sqrt{2gh}$

Example

A particle of mass m slides down the smooth inclined face of a wedge of mass $2m$, and inclination α, which is free to move in a smooth horizontal table. Use equations of momentum and energy to obtain an expression for the velocity of the particle relative to the wedge when the particle has moved a relative distance s from rest down the inclined face of the wedge.



Answer:

Let the velocity of the wedge be V and that of the particle relative to the wedge be u. By conservation of horizontal momentum, since there is no horizontal force acting on the system,

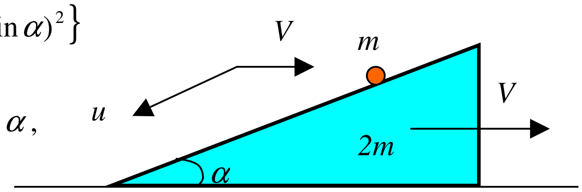
$2mV - m(u\cos\alpha - V) = 0.$

$\therefore \quad V = \frac{1}{3}u\cos\alpha.$

By conservation of energy,

$mgs\sin\alpha = \frac{1}{2}(2m)V^2 + \frac{1}{2}m\{(u\cos\alpha - V)^2 + (u\sin\alpha)^2\}$

$\therefore gs\sin\alpha = \frac{1}{9}u^2\cos^2\alpha + \frac{2}{9}u^2\cos^2\alpha + \frac{1}{2}u^2\sin^2\alpha,$



$$u = \left\{\frac{6gs\sin\alpha}{2+\sin^2\alpha}\right\}^{\frac{1}{2}}.$$

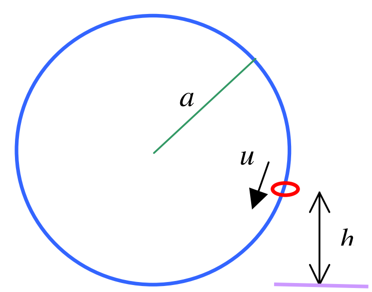
Example

A small ring of mass m can slide on a smooth circular wire of a radius a, which is fixed in a vertical plane. From a point on the wire at a vertical distance h above its lowest point the ring is given a velocity u along the wire. Find its velocity at the lowest point of the wire and determine the condition of u such that it could reach the highest point of the wire.

Answer:

Since there is no friction, the reaction between the ring and the wire is at any instant perpendicular to the wire, and therefore to the direction of motion of the ring. Hence this force does no work as the ring slides on the wire. Therefore, we may use the energy equation with the potential energy depending only on the height of the ring above the lowest point of the wire.

When the ring starts we have



Kinetic energy = $\frac{1}{2}mu^2$,

Potential energy = mgh,

Total energy = $\frac{1}{2}mu^2 + mgh$.

If v is the velocity at the lowest point of the wire, we have at that point

Kinetic energy = $\frac{1}{2}mv^2$,

Potential energy = 0,

Total energy = $\frac{1}{2}mv^2$.

Therefore

$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mgh,$$

$$v = \sqrt{u^2 + 2gh}.$$

Let w be the velocity at the highest point of the wire, we have in this position

Kinetic energy = $\frac{1}{2}mw^2$,

Potential energy = $mg \times 2a$,

Total energy = $\frac{1}{2}mw^2 + 2mga$.

Therefore

$$\frac{1}{2}mw^2 + 2mga = \frac{1}{2}mu^2 + mgh,$$

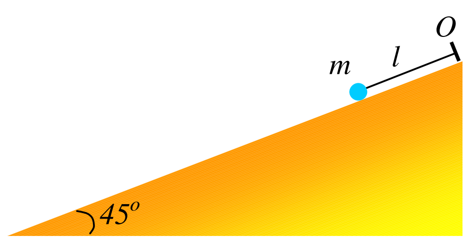
$$w^2 = u^2 + 2gh - 4ga.$$

If this expression is greater than zero, the ring will reach the highest point, and the condition for this is $u^2 \geq 2g(2a - h)$.

It should be noted that the direction along the wire of the initial velocity is immaterial to the problem.

Example

A particle of mass *m* is attached to one end of a light elastic string whose other end is fixed to a point *O* on an inclined plane (inclination angle = 45^o with the horizontal). The natural length of the string is *l* and its force constant is *mg/l*. The particle is held on the inclined plane so that the string lies just unstretched along a line of greatest slope and then released from rest.

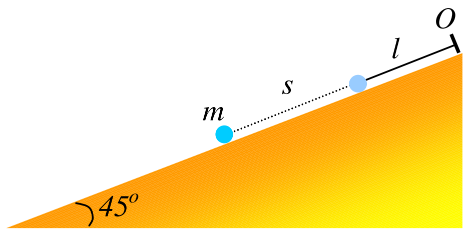


(a) Suppose that the inclined plane is smooth, determine the lowest position that the particle can reach. Determine also the equilibrium position.

(b) Suppose that the inclined plane is rough enough and the frictional coefficient is μ. Show that if the particle stops after its first descending a distance *d* along the greatest slope, then the distance is given by

$$d = \sqrt{2}l(1 - \mu).$$

Answer:



(a) Note that the particle is momentarily stopped at its lowest position. By the conservation of mechanical energy, the loss in gravitational energy becomes the gain in the elastic potential energy.

$$\frac{1}{2}\left(\frac{mg}{l}\right)s^2 = mg\,(s\sin 45^o)\,,$$

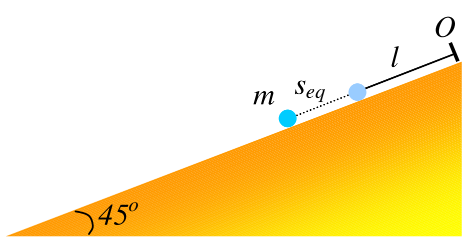
where s is the extension of the string when the particle is at its lowest position.

The above equation gives $\frac{1}{2}\left(\frac{s}{l}\right) = \frac{1}{\sqrt{2}}$. Hence, we obtain $s = \sqrt{2}l$. The lowest position is located at

$$l + s = l + \sqrt{2}l = (1+\sqrt{2})l \text{ from } O.$$

For equilibrium of the particle, the down plane force of the particle balances with the elastic force, we can write

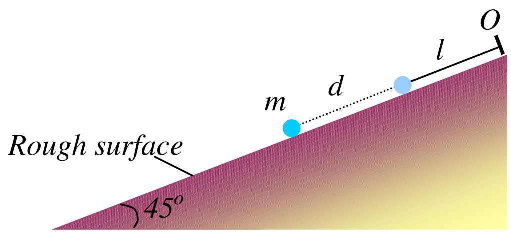
$$mg\sin 45^o = \left(\frac{mg}{l}\right)s_{eq}\,.$$



Hence, we obtain $s_{eq} = \frac{l}{\sqrt{2}}$. That is, the equilibrium position is located at

$$l + s_{eq} = l + \frac{l}{\sqrt{2}} = \left(1+\frac{1}{\sqrt{2}}\right)l \text{ from } O.$$

(b)



This part differs from part (a) by the frictional force. In the view of energy, we have

$$\frac{1}{2}(\frac{mg}{l})d^2 + \mu(mg\cos 45^o)\,d = mg\,(d\sin 45^o)\,.$$

After simplification, we have

$$\frac{1}{2}(\frac{d}{l}) + \frac{1}{\sqrt{2}}\mu = \frac{1}{\sqrt{2}}\,,$$

which gives $d = \sqrt{2}l(1-\mu)$.

Example

A bullet of mass m travelling horizontally with speed u strikes perpendicularly a wooden block of mass M which is free to move. The penetration is assumed uniform and the bullet comes to rest after penetrating a distance a into the block. Show that the ultimate common velocity of the system is $\frac{mu}{m+M}$ and that the total loss of kinetic energy during the penetration is $\frac{mMu^2}{2(m+M)}$. Deduce the value of the resistance and show that

(a) The block moves a distance $\frac{am}{m+M}$ before the bullet comes to rest relative to the block

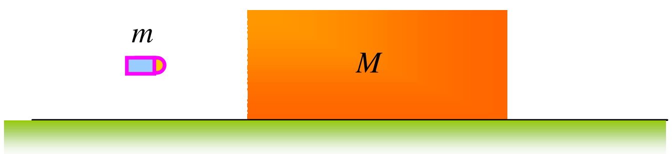
(b) The penetration by the bullet ceases after a time $\frac{2a}{u}$.

Answer:

By the conservation of momentum

$$mu = (M+m)v$$

$$\therefore \quad v = \frac{mu}{m+M}$$



The energy loss = $\frac{1}{2}mu^2 - \frac{1}{2}(M+m)v^2 = \frac{1}{2}mu^2 - \frac{1}{2}(M+m)(\frac{mu}{m+M})^2 = \frac{mMu^2}{2(m+M)}$

The energy loss = (resistance) · (displacement) = $Fa = \frac{mMu^2}{2(m+M)}$.

$\therefore \qquad F = \frac{mMu^2}{2a(m+M)}$.

(a) Let the displacement of the wooden block before the bullet comes to rest be x.

$$v^2 = 0^2 + 2(\frac{F}{M})x$$

$$(\frac{mu}{m+M})^2 = 2(\frac{mu^2}{2a(m+M)})x$$

$$x = \frac{am}{m+M}$$

(b) The time for block M to move a distance x = the time of penetration

$$x = (0)(t) + \frac{1}{2}(\frac{mu^2}{2a(m+M)})t^2$$

$\therefore \qquad \frac{am}{m+M} = \frac{1}{2}(\frac{mu^2}{2a(m+M)})t^2$

$\therefore \qquad t = \frac{2a}{u}$.