

Hong Kong Physics Olympiad Lesson 5

- 5.1 Circular Motion
- 5.2 Rotation with constant angular acceleration
- 5.3 Centripetal acceleration: $a_r = v^2/r$
- 5.4 The banked curve
- 5.5 The rotor

5.1 Circular Motion

For a rigid body to rotate around a fixed axis, each particle in the rigid body undergoes a circular motion. Take one of the particle as an example, θ is the angular displacement, s is the distance it moves and r is the distance between the axis and the particle.

$$\theta = \frac{s}{r}$$

- (a) The average angular velocity is defined as

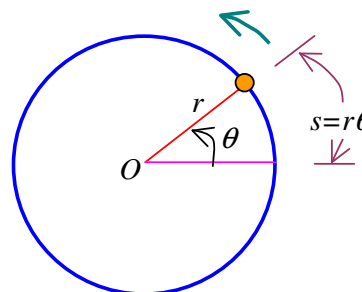
$$\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

- (b) Instantaneous angular velocity

$$\omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Remark:

In advanced study, ω is quite often defined as a vector with its direction pointed by the right-hand rule. This is common in the area of science and engineering.



- (c) Instantaneous tangential velocity

$$v_T = r\omega$$

- (d) Angular acceleration α

$$\alpha = \lim_{\Delta t \rightarrow 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

It is again a vector in advanced study!

- (e) Tangential acceleration

$$a_T = r\alpha$$

- Although every point in a rigid body travels different distance during the rotation, the angular displacements are the same. At any instant, every point in a rigid body has the same angular velocity.

5.2 Rotation with constant angular acceleration

$$(1) \quad \alpha = \frac{d\omega}{dt} = \text{constant}$$

$$\therefore \omega = \omega_0 + \alpha t \quad (v = v_0 + at)$$

$$\omega_0 = \omega(t)|_{t=0}$$

$$(2) \quad \omega = \frac{d\theta}{dt} \quad \therefore \int_{\theta_0}^{\theta} d\theta = \int_0^t \omega dt$$

$$\therefore \theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \quad (x = x_0 + v_0 t + \frac{1}{2} at^2)$$

$$= \theta_0 + \frac{1}{2} \omega_0 t + \frac{1}{2} \omega_0 t + \frac{1}{2} \alpha t^2$$

$$= \theta_0 + \frac{1}{2} \omega_0 t + \frac{1}{2} \omega t$$

$$= \theta_0 + \frac{1}{2} (\omega_0 + \omega) t$$

$$\therefore t = \frac{1}{\alpha} (\omega - \omega_0)$$

$$\therefore \theta - \theta_0 = \frac{1}{2} (\omega_0 + \omega) (\omega - \omega_0) \frac{1}{\alpha} \quad \therefore \omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0)$$

$$\boxed{\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)}$$

$$v^2 = v_0^2 + 2a(x - x_0)$$

For circular motion, $s = r\theta$ and r is constant.

$$\therefore \frac{ds}{dt} = r \frac{d\theta}{dt}$$

$$\frac{ds}{dt} = v \quad \text{Instantaneous speed}$$

$$\frac{d\theta}{dt} = \omega \quad \text{Angular velocity}$$

$$\therefore v = r\omega$$

For a non-uniform circular motion, the acceleration consists of two parts.

(1) Tangential component of acceleration

$$a_{\tan} \equiv \frac{dv}{dt} = r \frac{d\omega}{dt} = r\alpha \quad (\text{Non-uniform circular motion})$$

- Direction: always tangent to the circular path of the particle.

(2) Radial component of acceleration

$$a_{rad} = \frac{v^2}{r} = r\omega^2$$

5.3 Centripetal acceleration: $a_r = v^2/r$

Consider the case of uniform circular motion.

- Uniform circular motion: Speed is a constant, e.g.

$$|\vec{v}_1| = |\vec{v}_2| = v.$$

Obviously, triangles OP_1P_2 and Op_1p_2 are similar.

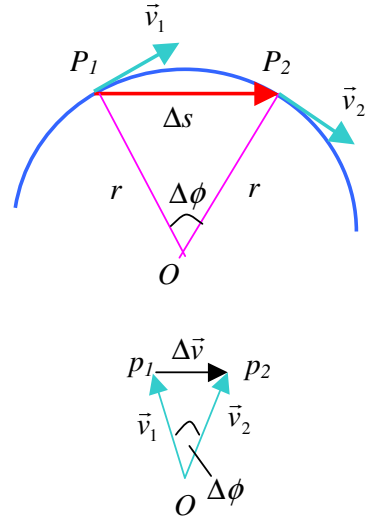
$$\therefore \frac{\Delta s}{r} = \frac{|\Delta \vec{v}|}{v_1} \quad \text{or} \quad |\Delta \vec{v}| = \frac{v_1}{r} \Delta s = \frac{v}{r} \Delta s$$

$$\text{Acceleration} \quad a_{av} = \frac{|\Delta \vec{v}|}{\Delta t} = \frac{v}{r} \frac{\Delta s}{\Delta t}$$

$$a = \lim_{\Delta t \rightarrow 0} a_{av} = \frac{v}{r} \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \frac{v^2}{r}$$

In fact, the centripetal acceleration is given by $\vec{a} = -\frac{v^2}{r} \hat{r}$, where \hat{r} is the unit vector pointed

from the origin. The acceleration can be rewritten as $\vec{a} = -\omega^2 r \hat{r}$, since $v = r\omega$.



Example

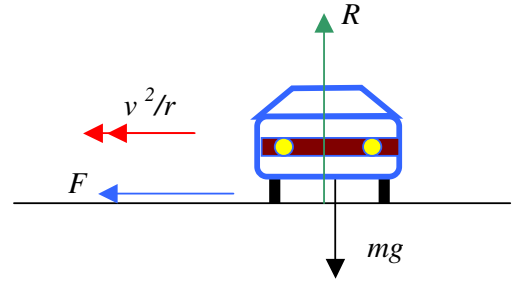
On a wet day the coefficient of friction between a car's tyres and the ground is 0.2. At what speed is it safe for a driver to round a corner of radius 25 m, if the road surface is horizontal?

Answer:

It is the frictional force F , which makes it possible for the car to move in a circle.

The equations of motions are

$$F = \frac{mv^2}{r} = \frac{mv^2}{25},$$
$$R = mg.$$



Since $F \leq \mu R$, we have

$$\frac{mv^2}{25} \leq 0.2 mg, \quad v^2 \leq 0.2(25)(10),$$

$$v^2 \leq 50, \quad v \leq 7.07 \text{ m/s}.$$

The maximum speed, if skidding is to be avoided, is 7.07 m/s.

5.4 The banked curve

A car of mass m travels at constant speed v round a bend of radius r on a road banked at an angle α . The coefficient of friction between the car's tyres and the road surface is $\tan \lambda$ where $\lambda < \alpha$. We have the following cases.

- (a) if the car travels with no tendency to slip,

$$v^2 = rg \tan \alpha,$$

- (b) if the car is about to slip outwards,

$$v^2 = rg \tan (\alpha + \lambda),$$

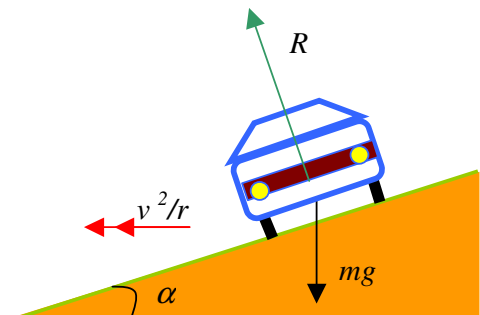
- (c) if the car is about to slip inwards,

$$v^2 = rg \tan (\alpha - \lambda).$$

Explanations:

- (a) In this case there is no frictional force acting.

The equation of motion are



$$R \sin \alpha = \frac{mv^2}{r}, R \cos \alpha - mg = 0.$$

Therefore, $v^2 = rg \tan \alpha$.

- (b) The frictional force acts down the slope and takes its limiting value, i.e. $F = \mu R$.

The equations of motion are

$$R \sin \alpha + \mu R \cos \alpha = \frac{mv^2}{r},$$

$$R \cos \alpha - \mu R \sin \alpha - mg = 0,$$

$$\text{Therefore, } v^2 = \frac{rR}{m}(\sin \alpha + \mu \cos \alpha),$$

$$R = \frac{mg}{\cos \alpha - \mu \sin \alpha}.$$

Hence when the car is about to slip outwards,

$$\begin{aligned} v^2 &= \frac{rg(\sin \alpha + \mu \cos \alpha)}{\cos \alpha - \mu \sin \alpha} \\ &= \frac{rg(\tan \alpha + \mu)}{1 - \mu \tan \alpha} = \frac{rg(\tan \alpha + \tan \lambda)}{1 - \tan \alpha \tan \lambda} = rg \tan (\alpha + \lambda). \end{aligned}$$

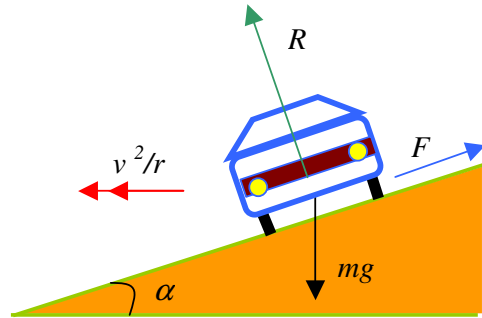
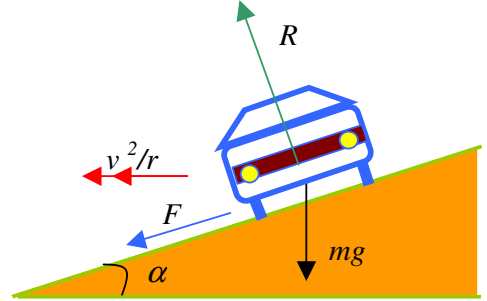
- (c) The frictional force acts up the slope and takes its limiting value μR . The equations of motion are

$$R \sin \alpha - \mu R \cos \alpha = \frac{mv^2}{r}$$

$$R \cos \alpha + \mu R \sin \alpha - mg = 0.$$

$$\text{Therefore, } v^2 = \frac{rg(\sin \alpha - \mu \cos \alpha)}{\cos \alpha + \mu \sin \alpha}$$

$$v^2 = \frac{rg(\tan \alpha - \tan \lambda)}{1 + \tan \alpha \tan \lambda} = rg \tan (\alpha - \lambda).$$



Example

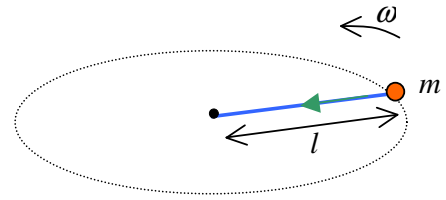
A particle of mass m is connected by an inextensible light string of length l to a fixed point on a smooth horizontal table. The string breaks when subject to a tension whose magnitude exceeds mg . Find the maximum number of revolutions per second that the particle can make without breaking the string.

Answer:

For maximum revolution ω , we have the maximum tension of string equals mg , i.e. $mg = m\omega^2 l$, which implies

$$\omega = \sqrt{\frac{g}{l}} \text{ rad/s}.$$

The frequency $f = \frac{1}{T} = \frac{1}{2\pi/\omega} = \frac{1}{2\pi} \sqrt{\frac{g}{l}} \text{ rev/s}.$



Example

A star is believed to rotate at about 1 rev/s. If such a star has a radius of 20 km (a typical value), (a) what is the speed of a point on the equator of the star and (b) what is the centripetal acceleration of this point?

Answer:

a) Angular velocity = 2π

$$\text{Speed} = 2\pi \times 20 = 125.7 \text{ km/s}$$

b) Centripetal acceleration = $(2\pi)^2 \times 20 = 789.6 \text{ km/s}^2$

$$\text{Centripetal acceleration} \quad a = \frac{v^2}{R}$$

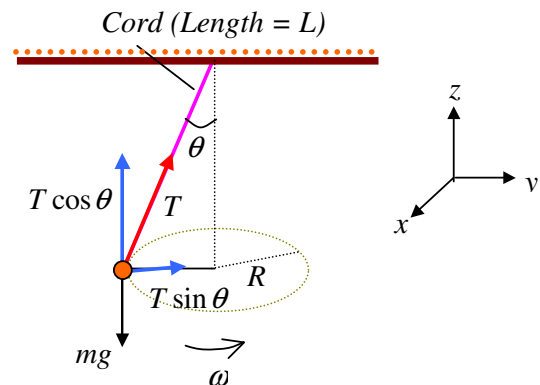
$$\therefore F = ma = \frac{mv^2}{R}$$

Example

Find the period of the conical pendulum as shown in figure.

Answer:

Period: The time required to make a complete revolution.



- The particle moves in a plane x - y , so there is no motion along z -direction

$$\sum F_z = 0 \quad \text{or} \quad T \cos \theta - mg = 0 \Rightarrow T = \frac{mg}{\cos \theta}$$

- In x - y plane, it is a uniform circular motion

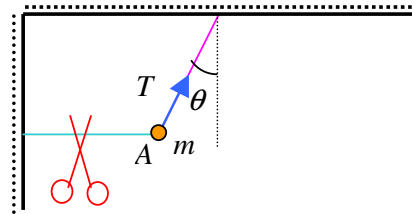
$$F = \frac{mv^2}{R} \quad \text{or} \quad T \sin \theta = \frac{mv^2}{L \sin \theta}$$

$$\text{The period } \tau = \frac{2\pi R}{v} = \frac{2\pi L \sin \theta}{v} = 2\pi \sqrt{\frac{L \cos \theta}{g}}.$$

Example

Find the tension T before and after the string is cut.

Answer:

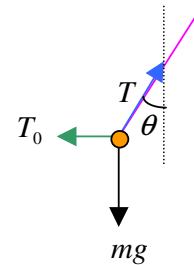


- (a) Before the cut, the system is in equilibrium

$$\Sigma \mathbf{F} = 0$$

$$\begin{aligned} \sum F_x = 0 &\Rightarrow T_0 = T \sin \theta \\ \sum F_y = 0 &\Rightarrow mg = T \cos \theta \end{aligned}$$

Hence, we have $T = \frac{mg}{\cos \theta}$.



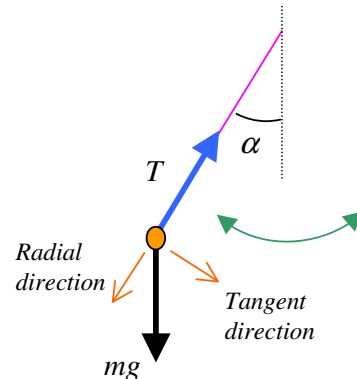
- (b) Once the string is cut, it is not in equilibrium anymore. It will undergo a circular motion (non-uniform).

For circular motion, say, it is at an angle α with the vertical.

$$\text{Resultant force along radial direction} = m \frac{v^2}{R}.$$

$$\text{Radial acceleration: } T - mg \cos \alpha = m \frac{v^2}{R} = ma_r,$$

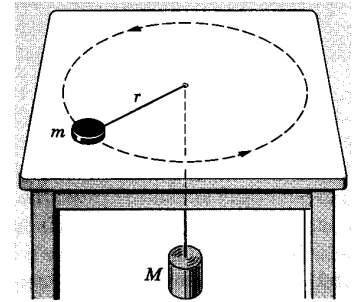
$$\text{Or } T = mg \cos \alpha + ma_r.$$



Tangential acceleration: $ma_t = mg \sin \alpha$.

Example

A disk of mass m on a frictionless table is attached to a hanging cylinder of mass M by a cord through a hole in the table (see figure). Find the speed with which the disk must move in a circle of radius r for the cylinder to stay at rest.

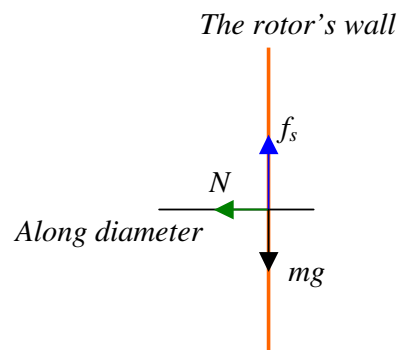
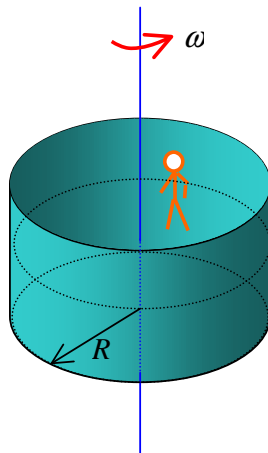


Answer:

For M to remain at rest the tension T in the cord must equal the gravitational force Mg exerted on M . The tension supplies the centripetal force that keeps m in its circular orbit, so $T = mv^2/r$. Thus $Mg = mv^2/r$ and we obtain $v = \sqrt{Mgr/m}$.

5.5 The Rotor

Rotor is quite often found in amusement park for visitors to enjoy with. A rotor is a hollow cylindrical room that can be set to rotate about the central vertical axis. A person enters the rotor and stand against the wall. The rotor gradually increases its rotating speed up to a preset one and the floor below the person is opened downward. The person does not fall down but remains rotating with the rotor.



The man rotates with the rotor and the centripetal force which acts on him is provided by the wall as normal force, $N = \frac{mv^2}{R}$. As the man does not fall down, the frictional force in the upward direction balance with his weight, e.g. $f_s = mg = \mu_s N$. Hence, we obtain

$$mg = \mu_s \left(\frac{mv^2}{R} \right) \text{ and thus } v = \sqrt{\frac{gR}{\mu_s}}.$$

It should be noted that the velocity does not depend on the mass of the man. A practical data for the coefficient of friction between the textile material of clothing and a rotor wall is about 0.40. For a rotor of radius of 2.0 m, v is about 7.0 m/s or more. That is the frequency f is given by

$$f = \frac{7.0 \text{ m/s}}{2\pi(2.0 \text{ m})} = 0.56 \text{ revolution/s or about 33 rpm.}$$

Example

A bucket of water can be swung round in a vertical circle without falling spilling. If the weight mg of the water is less than mv^2/r , the normal force N of the bottom of the bucket on the water provides the rest of the force required to maintain the water in its circular path. However, if the bucket is swung more slowly then mg will be greater than mv^2/r and the ‘unused’ part of the weight causes the water to leave the bucket.

Consider the bucket in position A, the circular motion is maintained by the centripetal force which is contributed by the normal reaction N and the weight of water mg , i.e.

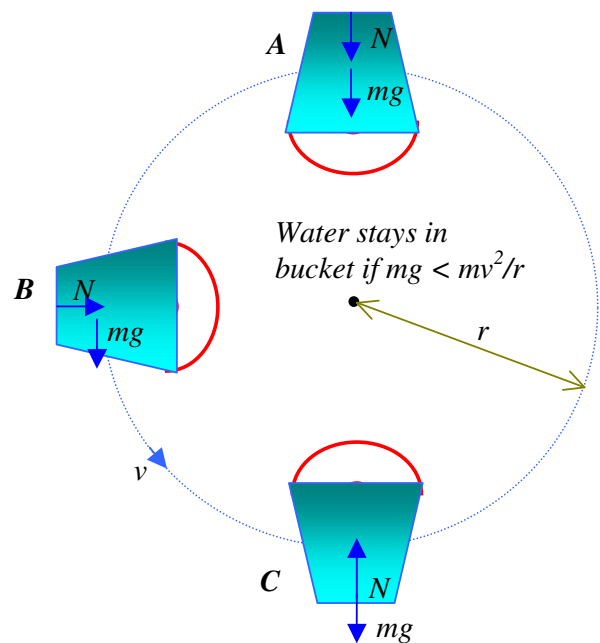
$$N + mg = \frac{mv^2}{r},$$

if water is still in the bucket, $N > 0$, and

$$mg < \frac{mv^2}{r}.$$

When the bucket is in position B, the centripetal force is provided solely by the normal reaction N , i.e.

$$N = \frac{mv^2}{r}.$$

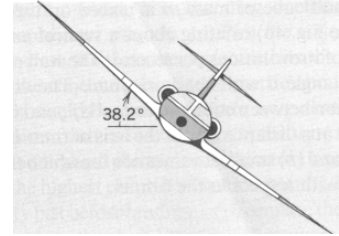


Lastly, when the bucket is in position C, the difference of the normal reaction N and the weight of water mg contributes to the centripetal force, i.e.

$$N - mg = \frac{mv^2}{r}.$$

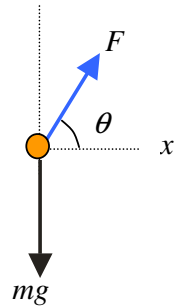
Example

An aeroplane is flying in a horizontal circle at a speed of 482 km/h., the wings of the plane are tilted at 38.2° to the horizontal; see figure. Find the radius of the circle in which the plane is flying. Assume that the centripetal force is provided entirely by the lift force perpendicular to the wing surface.



Answer:

The free-body diagram for the plane is shown in the figure. F is the magnitude of the lift on the wings and m is the mass of the plane. Since the wings are tilted 38.2° to the horizontal, the angle θ is 51.8° . The center of the circular orbit is to the right of the plane, the dotted line being a portion of the radius. Take the x -axis to be to the right and the y -axis to be upward. Then the x component of Newton's second law is $F \cos \theta = mv^2/R$ and the y component is $F \sin \theta - mg = 0$, where R is the radius of the orbit. The first equation gives $F = mv^2/R \cos \theta$ and when this is substituted into the second equation, $(mv^2/R) \tan \theta = mg$ results. Hence we can write



$$R = \frac{v^2}{g} \tan \theta.$$

The speed of the plane is $v = 482 \text{ km/h} = 133.8 \text{ m/s}$, so

$$R = \frac{133.8^2}{9.8} \tan 51.8^\circ = 2.32 \times 10^3 \text{ m}$$

Example

A very small cube of mass m is placed on the inside of a funnel (see figure) rotating about a vertical axis at a constant rate of ν revolutions per second. The wall of the funnel makes an angle θ with the horizontal. The coefficient of static friction between cube and funnel is μ_s

and the center of the cube is at a distance r from the axis of rotation. Find the (a) smallest and (b) largest values of v for which the cube will not move with respect to the funnel.

Answer:

a, b) Angular velocity of the funnel $= 2\pi v$

$$\text{Centripetal force} = m(2\pi v)^2 r$$

Maximum frictional force $f_{\max} = \mu_s N$, where N is normal reaction.

Case (a), for smallest revolutions, the frictional force should point upwards along the wedge.

Equation of forces along x -axis:

$$N \sin \theta - f_{\max} \cos \theta = m(2\pi v)^2 r$$

$$N \sin \theta - \mu_s N \cos \theta = m(2\pi v)^2 r$$

$$N = \frac{m(2\pi v)^2 r}{\sin \theta - \mu_s \cos \theta} \quad (1)$$

Equation of forces along y -axis:

$$N \cos \theta + f_{\max} \sin \theta = mg$$

$$N(\cos \theta + \mu_s \sin \theta) = mg \quad (2)$$

Combine equation (1) and (2)

$$\frac{m(2\pi v)^2 r(\cos \theta + \mu_s \sin \theta)}{(\sin \theta - \mu_s \cos \theta)} = mg$$

$$v = \frac{1}{2\pi} \sqrt{\frac{g(\sin \theta - \mu_s \cos \theta)}{r(\cos \theta + \mu_s \sin \theta)}}$$

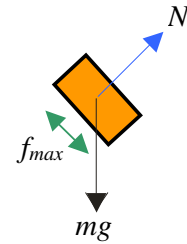
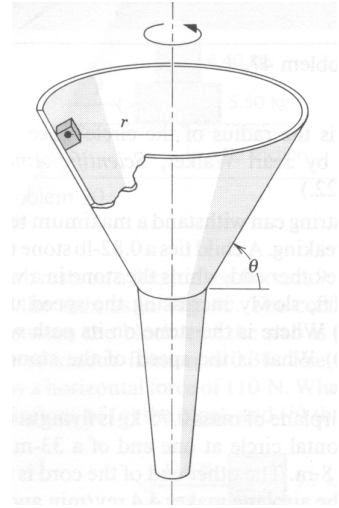
$$v = \frac{1}{2\pi} \sqrt{\frac{g(\tan \theta - \mu_s)}{r(1 + \mu_s \tan \theta)}}$$

Case (b), for largest revolutions, the frictional force will point downwards along the wedge.

Equation of force along x -axis:

$$N \sin \theta + f_{\max} \cos \theta = m(2\pi v)^2 r$$

$$N \sin \theta + \mu_s N \cos \theta = m(2\pi v)^2 r$$



$$N = \frac{m(2\pi\nu)^2 r}{\sin\theta + \mu_s \cos\theta} \quad (3)$$

Equation of force along y-axis:

$$\begin{aligned} N \cos\theta - f_{\max} \sin\theta &= mg \\ N(\cos\theta - \mu_s \sin\theta) &= mg \end{aligned} \quad (4)$$

Combine equation (3) and (4), we have

$$\begin{aligned} \frac{m(2\pi\nu)^2 r(\cos\theta - \mu_s \sin\theta)}{(\sin\theta + \mu_s \cos\theta)} &= mg \\ \nu &= \frac{1}{2\pi} \sqrt{\frac{g(\sin\theta + \mu_s \cos\theta)}{r(\cos\theta - \mu_s \sin\theta)}} \\ \nu &= \frac{1}{2\pi} \sqrt{\frac{g(\tan\theta + \mu_s)}{r(1 - \mu_s \tan\theta)}} \end{aligned}$$