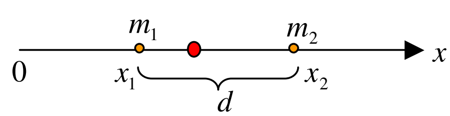
Hong Kong Physics Olympiad Lesson 6

6.1 System of particles

(1) Two particles in 1-D



Total mass: $M = m_1 + m_2$

Center of mass: $x_{CM} \equiv \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \dfrac{m_1 x_1 + m_2 x_2}{M}$

If $m_1 = m_2 = m$ then $x_{CM} = \dfrac{m(x_1 + x_2)}{2m} = \dfrac{x_1 + x_2}{2}$

(2) N particles in 1-D

$$x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2 + \cdots\cdots + m_N x_N}{m_1 + m_2 + m_3 + \cdots\cdots + m_N} = \frac{\sum_{i=1}^{N} m_i x_i}{M}$$

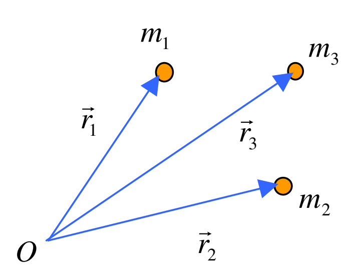
(3) N particles in 3-D

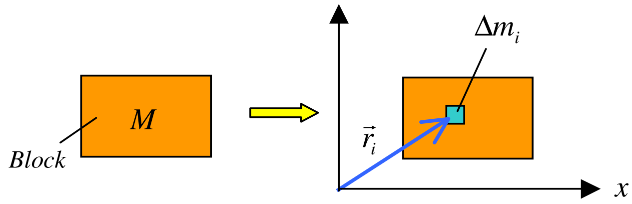
$\boldsymbol{r}_1 = (x_1, y_1, z_1)$

$\boldsymbol{r}_2 = (x_2, y_2, z_2)$

$\cdots\cdots$

$$\begin{cases} x_{CM} = \dfrac{1}{M}\displaystyle\sum_{i=1}^{N} m_i x_i \\ y_{CM} = \dfrac{1}{M}\displaystyle\sum_{i=1}^{N} m_i y_i \\ z_{CM} = \dfrac{1}{M}\displaystyle\sum_{i=1}^{N} m_i z_i \end{cases} \Rightarrow \boxed{\boldsymbol{r}_{CM} = \frac{1}{M}\sum_{i=1}^{N} m_i \boldsymbol{r}_i}$$





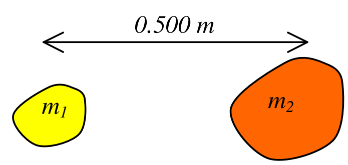
$$x_{CM} = \frac{1}{M}\sum_i x_i \Delta m_i \qquad \text{When } \Delta m_i \to 0 \qquad x_{CM} = \frac{1}{M}\int x\,dm$$

$$y_{CM} = \frac{1}{M}\sum_i y_i \Delta m_i \qquad\qquad y_{CM} = \frac{1}{M}\int y\,dm$$

In general, $$\boxed{\mathbf{r}_{CM} = \frac{1}{M}\int \mathbf{r}\,dm}$$

Example

Suppose the masses in the following figure are separated by 0.500 m, and that m_1 = 0.260 kg and m_2 = 0.170 kg. What is the distance from m_1 to the center of mass of the system?



Answer:

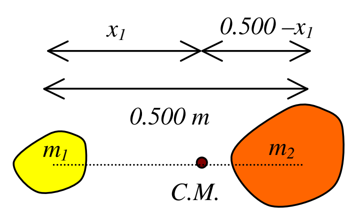
Method 1:

We know that the center of mass lies on the line joining m_1 and m_2. Now, let O be the center of mass and x_1 be the distance between m_1 and O. We can write

$$x_1 m_1 = (0.500 - x_1) m_2$$

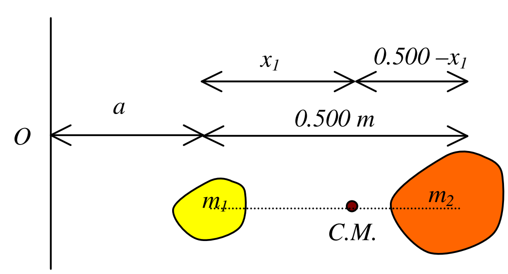
That is, $x_1(0.260) = (0.500 - x_1)(0.170)$, and we obtain $x_1 = 0.198\,\text{m}$.

This is the location of the center of mass.

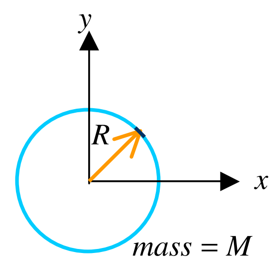


Method 2:

We take an arbitrary point O as the origin which has a distance a from m_1. Hence, we can write $a + x_1 = \dfrac{m_1 a + m_2(a + 0.500)}{m_1 + m_2}$, which gives $x_1 = 0.198$. The position of the C.M. is 0.198 m from m_1.



Example



Find the center of mass of a ring.

Answer:

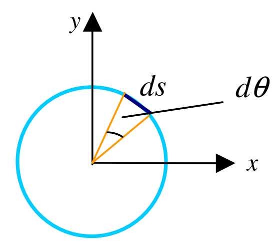
$$x_{CM} = \frac{1}{M}\int x dm, \qquad \text{Linear mass density } = \frac{M}{2\pi R} = \lambda$$

$$dm = ds \cdot \lambda \qquad \therefore\ M = \int dm = \lambda \int ds = \lambda \cdot 2\pi R = M$$

$$\begin{aligned} x &= R\cos\theta \\ y &= R\sin\theta \end{aligned} \qquad 0 \le \theta \le 2\pi, \quad ds = Rd\theta \qquad \text{Check } \int ds = R\int d\theta \Rightarrow 2\pi R = R \cdot 2\pi$$

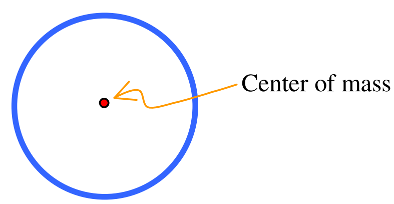
$$\begin{aligned} x_{CM} &= \frac{1}{M}\int_0^{2\pi} R\cos\theta \cdot R\lambda d\theta \\ &= \frac{R^2}{M}\lambda \int_0^{2\pi} \cos\theta\, d\theta \\ &= \frac{R^2}{M}\lambda \sin\theta \Big|_0^{2\pi} = 0 \end{aligned}$$

$$\int \cos\theta\, d\theta = \sin\theta$$



$$\begin{aligned} y_{CM} &= \frac{1}{M}\int y\, dm \\ &= \frac{1}{M}\int_0^{2\pi} R\sin\theta \cdot R\lambda d\theta \\ &= \frac{R^2}{M}\lambda \int_0^{2\pi} \sin\theta d\theta \\ &= -\frac{R^2}{M}\lambda \cos\theta \Big|_0^{2\pi} = 0 \end{aligned}$$

$$\int \sin\theta\, d\theta = -\cos\theta$$



$\therefore \qquad \boldsymbol{r}_{CM} = 0$

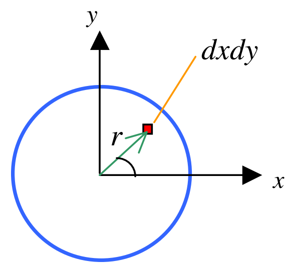
Example

Find the center of mass of a disk.

Answer:

A disk can be considered as a system of rings. Hence the center of mass of a disk is at the center of the disk.

Or, one can use direct integration.



$$x_{CM} = \frac{1}{M}\int x dm, \qquad \text{Density } = \frac{M}{\pi R^2} = \rho$$

$$dm = \rho dxdy$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad dxdy = rdrd\theta$$

$$x_{CM} = \frac{1}{M}\int_0^R \int_0^{2\pi} r\cos\theta \rho r dr d\theta$$

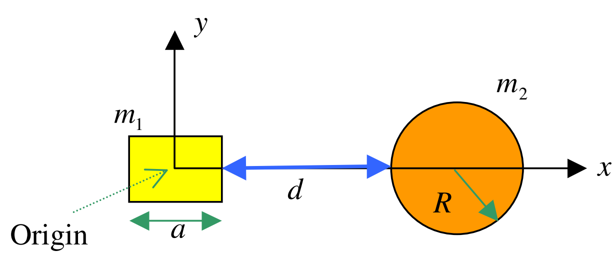
$$= \frac{\rho}{M}\int_0^R r^2 dr \int_0^{2\pi} \cos\theta d\theta = 0$$

$$y_{CM} = \frac{1}{M}\int_0^R \int_0^{2\pi} r\sin\theta \rho r dr d\theta$$

$$= \frac{\rho}{M}\int_0^R r^2 dr \int_0^{2\pi} \sin\theta d\theta = 0$$

Example

You are given two rigid bodies, one block and one disk. Find the center of mass of this system.



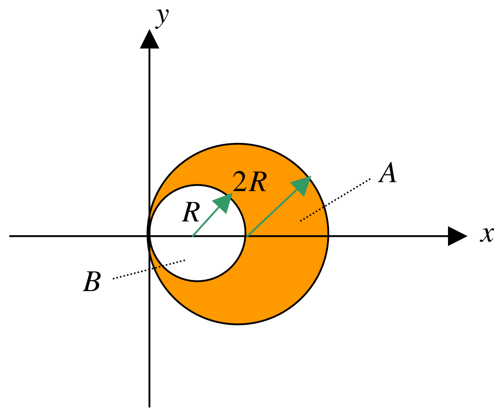
Answer:

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_2(d + R + \frac{a}{2})}{m_1 + m_2}$$

$$y_{CM} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = 0$$

Example

Find the center of mass of a special disk. Note that a part of it is empty.



Answer:

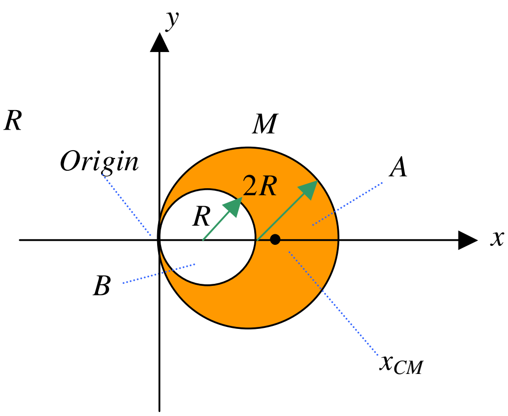
The disk of radius 2*R* consists of two parts *A* and *B,* namely, the larger disk *A,* and the smaller disk *B* (empty). The center of mass of the large disk is at $x_{CA} = 2R$, and $y_{CA} = 0$. The center of mass of the smaller disk is at $x_{CB} = R$, and $y_{CB} = 0$. Note that the mass of the smaller disk is defined as negative.

$$x_{CM} = \frac{m_A x_{CA} - m_B x_{CB}}{m_A - m_B} = \frac{m_A(2R) - m_B(R)}{m_A - m_B} = \frac{2m_A - m_B}{m_A - m_B} R$$

Define the mass density ρ = mass per unit area

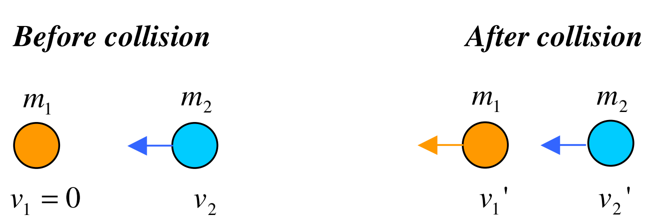
$$x_{CM} = \frac{\rho[2(4\pi R^2) - \pi R^2]}{\rho[4\pi R^2 - \pi R^2]} R$$

$$\therefore x_{CM} = \frac{7}{3} R$$



That is, the center of mass is *R*/3, from the center of the disk, as shown in figure.

6.2 Collision and the Newton's second law for a system of particles



If you are at the center of mass frame, you will see that the velocity of center of mass

$$\boldsymbol{v}_{CM} = \frac{d\boldsymbol{r}_{CM}}{dt} = \frac{d}{dt}\left(\frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2}\right) = \frac{m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2}{m_1 + m_2}$$ Does not change! (Why?)

For a system of particles

Internal forces

$$M\boldsymbol{a}_{CM} = \sum \boldsymbol{F}_{ext}$$

- $\boldsymbol{F}_{ext}$: Vector sum of all external forces.
- M: Total mass of the system
- $\boldsymbol{a}_{CM}$: Acceleration of the center of mass

e.g. For a two-particles system: $\boldsymbol{a}_{CM} = \dfrac{m_1\boldsymbol{a}_1 + m_2\boldsymbol{a}_2}{m_1 + m_2} = \dfrac{d^2\boldsymbol{r}_{CM}}{dt^2}$

- There are three equations for a description of 3-D system, each for different direction *x, y,* and *z, e.g.*

$$\boldsymbol{r}_{CM} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 + \cdots\cdots + m_n\boldsymbol{r}_n}{M}, \qquad M = m_1 + m_2 + \cdots\cdots + m_n$$

$$\therefore M\boldsymbol{r}_{CM} = m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 + \cdots\cdots + m_n\boldsymbol{r}_n$$

Taking the second derivative with respect to t,

$$\begin{aligned} M\boldsymbol{a}_{CM} &= m_1\boldsymbol{a}_1 + m_2\boldsymbol{a}_2 + \cdots\cdots + m_n\boldsymbol{a}_\mathrm{n} \\ &= \boldsymbol{F}_1 + \boldsymbol{F}_2 + \cdots\cdots + \boldsymbol{F}_n, \end{aligned}$$

where $\boldsymbol{F}_i$ is the total forces acting on particle i.

$$\boldsymbol{F}_i = \boldsymbol{F}_{i(ext)} + \boldsymbol{F}_{i(\mathrm{int})}$$

$\boldsymbol{F}_{i(\mathrm{int})} \Rightarrow$ The forces exerted by the particle within the system. *e.g.* action force or reaction force.

As the action force and the reaction force come in pairs. We have

$$\sum_i F_{i(int)} = 0$$

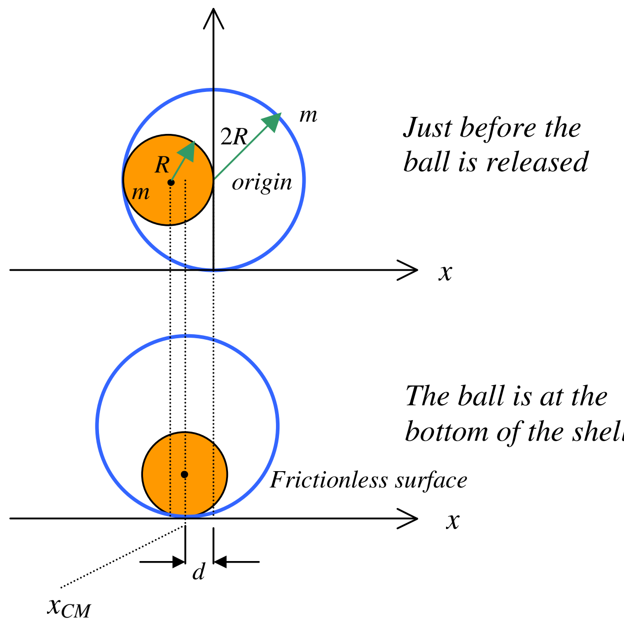
$$\therefore M\boldsymbol{a}_{CM} = \sum \boldsymbol{F}_{ext}$$

When a body or a collection of particles is acted on by external forces, the center of mass moves just as all the masses were concentrated at that point and it is acted on by a resultant force which equals to the sum of the external forces on the system.

Example

Given a spherical shell of mass m and its inner radius $2R$. A ball of mass m and radius R is released as shown in figure. The ball will roll back and forth. What will be the displacement of the shell when the ball is at the bottom of the shell?

Answer:



From Newton's second law

$$\sum \boldsymbol{F}_{ext} = 2m\boldsymbol{g} + \boldsymbol{Normal\ Reaction} = 2m\boldsymbol{a}_{CM}$$

Along x-direction $\quad \sum F_{ext,x} = 0$

$$\therefore a_{ext,x} = 0, \quad a = \frac{dv}{dt}$$

$$\therefore v_{CM} = \text{constant}$$

Originally, $v_{CM} = 0$, $\therefore$ it remains $\quad v = \dfrac{dx}{dt}$

$$\therefore x_{CM} = \text{constant}$$

Before release,

$$x_{CM} = \frac{m \cdot 0 + m(-R)}{2m} = -\frac{R}{2}$$

When the ball is at the bottom of the shell, $\quad x_{CM} = \dfrac{m(-d) + m(-d)}{2m} = -d$.

Since there are no external forces acting on the system horizontally, hence x_{CM} does not move, that is $x_{CM} = -R/2 = -d$. Or, we can say the shell moves $R/2$ to the left from its original position.

6.3 The momentum of a system of particles

- For a one particle system

  $\boldsymbol{p} = m\boldsymbol{v}$ $\qquad$ $\boldsymbol{p}$**:** momentum of a particle, it is a vector.

- Newton's second law for a single particle

  $$\sum \boldsymbol{F} = m\frac{d\boldsymbol{v}}{dt} = \frac{d\boldsymbol{p}}{dt} \qquad \text{Rate of change of momentum}$$

- The momentum of a system of particles

$$\boldsymbol{P} = \sum_{i=1}^{N} \boldsymbol{p}_i = \sum_{i=1}^{N} m_i \boldsymbol{v}_i .$$

Remember that $$\boldsymbol{v}_{CM} = \frac{\sum_{i=0}^{N} m_i \boldsymbol{v}_i}{\sum_{i=0}^{N} m_i} = \frac{\sum_{i=0}^{N} m_i \boldsymbol{v}_i}{M}$$

$$\therefore \boxed{\boldsymbol{P} = M\boldsymbol{v}_{CM}}$$

- Newton's second law for a system of particles

$$\sum \boldsymbol{F}_{ext} = M\boldsymbol{a}_{CM} = M\frac{d\boldsymbol{v}_{CM}}{dt} = \frac{d\boldsymbol{P}}{dt}$$

If $\sum \boldsymbol{F}_{ext} = 0 \Rightarrow \frac{d\boldsymbol{P}}{dt} = 0$, or $\boldsymbol{P}$ = constant (law of conservation of momentum)

$\boldsymbol{P}$ = constant, $\quad \therefore \boldsymbol{v}_{CM}$ = constant

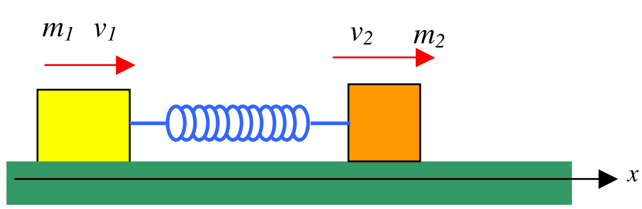
Example

The blocks are pulled apart and then released from rest.

(a) Discuss the center of mass of the system after release.

(b) What fraction the total kinetic energy of the system will each block occupy at any later time?

At any later time after release



Answer:

Since, there are no external forces acting horizontally, the motions of the blocks are due to internal force (i.e. the restoring force of the spring), the center of mass does not change.

$\sum F_{ext,x} = 0$, $\therefore P_x$ is conserved.

At $t = 0$, $P_x = 0$

At any later time: $P_x = m_1 v_1 + m_2 v_2 = 0$

$$\therefore \frac{v_1}{v_2} = -\frac{m_2}{m_1},$$

$$f_1 = \frac{K_1}{K_1 + K_2} = \frac{\frac{1}{2}m_1 v_1^2}{\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2} = \frac{1}{1 + \frac{m_2}{m_1}\frac{v_2^2}{v_1^2}}$$

$$= \frac{1}{1 + \frac{m_2}{m_1}(\frac{m_1^2}{m_2^2})} = \frac{1}{1 + \frac{m_1}{m_2}} = \frac{m_2}{m_1 + m_2}$$

$$f_2 = \frac{K_2}{K_1 + K_2} = \frac{1}{\frac{K_1}{K_2} + 1} = \frac{1}{\frac{m_1 v_1^2}{m_2 v_2^2} + 1} = \frac{1}{\frac{m_2}{m_1} + 1} = \frac{m_1}{m_1 + m_2}$$

$$f_1 + f_2 = \frac{m_2}{m_1 + m_2} + \frac{m_1}{m_1 + m_2} = 1 .$$

6.4 Appendix – Centers of gravity of uniform bodies

Uniform body	**Position of center of gravity on the axis of symmetry**
Solid tetrahedron, pyramid, cone	$\frac{h}{4}$ from base
Hollow and without base tetrahedron, pyramid, cone	$\frac{h}{3}$ from base
Arc subtending an angle 2α at center	$\frac{r \sin \alpha}{\alpha}$ from center
Semi-circular arc	$\frac{2r}{\pi}$ from center
Circular sector subtending an angle 2α at the center	$\frac{2r \sin \alpha}{3\alpha}$ from center
Semi-circle	$\frac{4r}{3\pi}$ from center
Solid hemisphere	$\frac{3r}{8}$ from plane face
Hollow without base hemisphere	$\frac{r}{2}$ from plane face