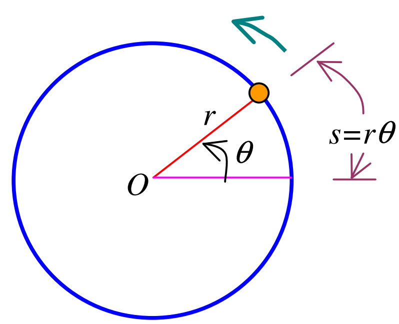
Hong Kong Physics Olympiad Lesson 7

7.1 Rotational kinematics



- Definitions of some useful quantities

(a) Angular position θ = Angle measured from the reference line

SI Unit: radian, which is dimensionless.

$\theta > 0$ counterclockwise rotation from reference line

$\theta < 0$ clockwise rotation from reference line

1 revolution = $360^{\circ} = 2\pi$ rad 1 rad ~57.3°

(b) Instantaneous velocity ω = Rate of change of angular displacement

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$ SI Unit: rad / s

$\omega > 0$ counterclockwise rotation

$\omega < 0$ clockwise rotation

(c) Period T = The time to complete one revolution

$$T = \frac{2\pi}{\omega} \qquad \text{SI Unit: second, s}$$

(d) Frequency f = Number of oscillation per second.

$$f = \frac{1}{T} \qquad \text{SI Unit: s}^{-1}$$

(e) Angular acceleration α = Rate of change of angular velocity

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} \qquad \text{SI Unit: s}^{-2}$$

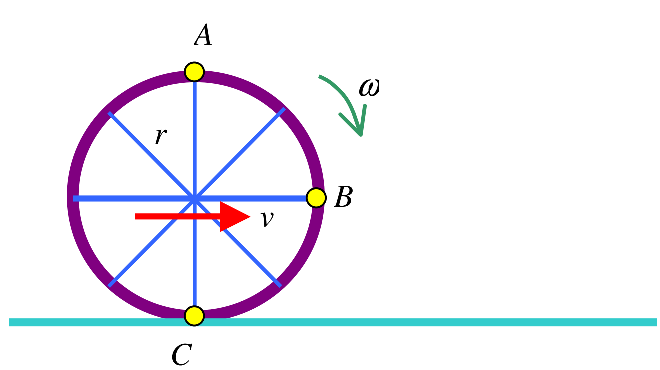
7.2 A comparison of linear kinematics and rotational kinematics

Linear Quantity	Angular Quantity
x	θ
v	ω
a	α

Linear Equation	Angular Equation
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

Example

Find the speeds of points *A, B* and *C* at the wheel, if the wheel is rotating without slipping with a uniform speed ω on the horizontal plane.



Answer:

The speed of the wheel is

$$v = \omega r.$$

The speed of point A: $v_A = \omega r + \omega r = 2\omega r.$

The speed of point B: $v_B = \sqrt{(r\omega)^2 + (r\omega)^2} = \sqrt{2} r\omega$.

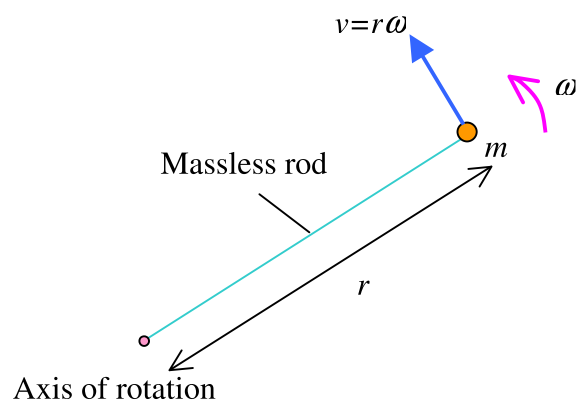
The speed of point C: $v_C = \omega r - \omega r = 0.$

7.3 Rotational kinetic energy and moment of inertia

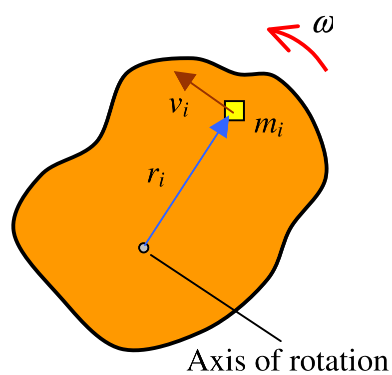
The kinetic energy of the rotating particle is given by

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(r\omega)^2 = \frac{1}{2}(mr^2)\omega^2 = \frac{1}{2}I\omega^2$$

where $I = mr^2$ is defined as the moment of inertia.



For a rigid body, there are many particles rotating at the same time with the same angular



velocity ω, the kinetic energy of it is given by

$$K = \sum_i K_i = \sum_i \frac{1}{2}m_i v_i^2 = \frac{1}{2}\sum_i m_i (r_i\omega)^2 = \frac{1}{2}\sum_i (m_i r_i^{\,2})\omega^2 = \frac{1}{2}I\omega^2$$

where $I = \sum_i m_i r_i^2$ is the moment of inertia of a rigid body.

Rigid Objects of Various Shapes	Axis of Rotation	Moment of Inertia
Ring or cylindrical hollow	Along the axis of cylinder	$I = MR^2$
Disk or solid cylinder	Along the axis of cylinder	$I = \frac{1}{2}MR^2$
Hollow sphere	Along the axis of sphere	$I = \frac{2}{3}MR^2$
Solid sphere	Along the axis of sphere	$I = \frac{2}{5}MR^2$
Long thin rod	Axis through the center of rod	$I = \frac{1}{12}ML^2$
Long thin rod	Axis through the rim of rod	$I = \frac{1}{3}ML^2$
Solid plate (L: length of plate)	Axis through center, in plane of plate	$I = \frac{1}{12}ML^2$
Solid plate (L: length of plate, W: width of plate)	Axis through center, perpendicular of plane of plate	$I = \frac{1}{12}M(L^2 + W^2)$

If a disk rotates without slipping on a frictionless plane, then the kinetic energy K contains two parts: the translational kinetic energy K_{trans} and the rotational kinetic energy K_{rot}.

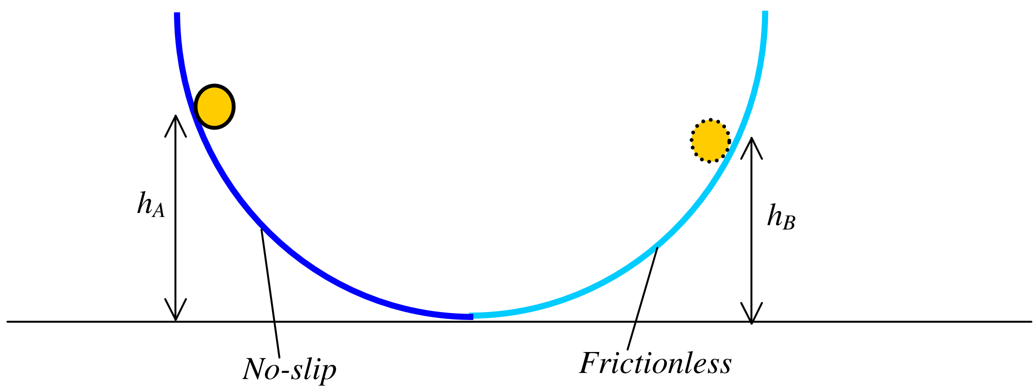
$$K = K_{trans} + K_{rot} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

In the above relation, $v = r\omega$ and I is the moment of inertia of the disk.

Example

A ball is released from rest at a height h_A on a no-slip surface. After reaching its lowest point, the ball begins to rise again, this time on a frictionless surface, and climb up to a height h_B. Compare the heights h_A and h_B.

Answer:



As the region in the left hand side gives no slip to the ball, it should be a rough region. The velocity of ball relates to its angular velocity by the relation: $v = r\,\omega$.

By the conservation of energy, we have

$$mgh_A = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 .$$

As the curved region in the right hand side is frictionless, the angular velocity does not change and the rotational kinetic energy is kept constant. The translational kinetic energy is used to overcome the gravitation. Hence, we have

$$mgh_B = \frac{1}{2}mv^2 .$$

Or, we can conclude that $h_B < h_A$.

7.4 Torque and angular acceleration

Consider a mass m connected to an axis of rotation by a light rod of length r. A tangential force of magnitude F is applied to the mass, and the mass is then rotated about the axis. According to Newton's second law, the tangential acceleration a is given by

$$a = \frac{F}{m} .$$

As the angular acceleration α relates the tangential acceleration a by

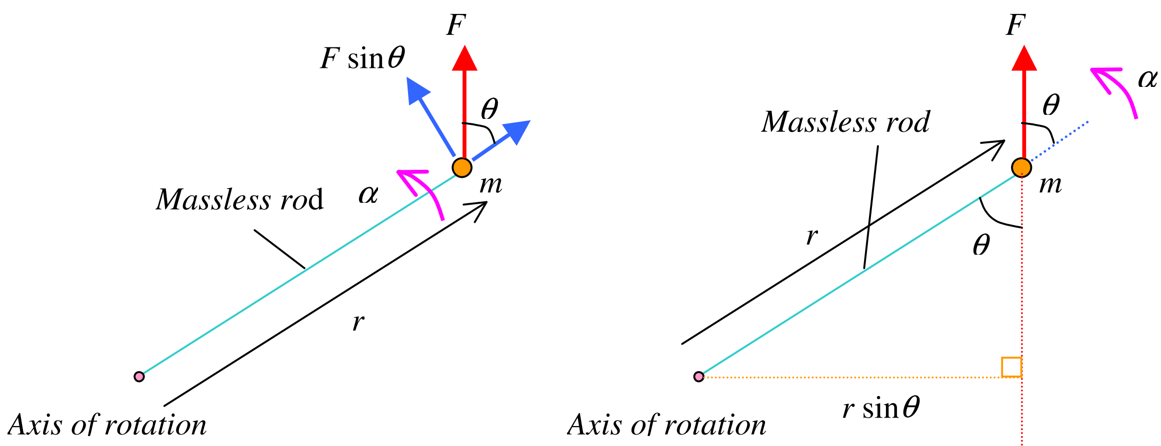
$$\alpha = \frac{a}{r}.$$

We have

$$\alpha = \frac{a}{r} = \frac{F}{mr} = \frac{rF}{mr^2} = \frac{\tau}{I}.$$

That is, $\tau = I\alpha$, where $\tau = rF$. The unit of torque is *Nm*. Note that *F* and *r* are perpendicular to each other. In general, if *F* makes an angle θ with *r*, the component of *F* which is perpendicular to *r* will be account to the torque, e.g.

$$\tau = r(F\sin\theta).$$



Or, we can rewrite the relation again as $\tau = (r\sin\theta)F$, where $r\sin\theta$ is the perpendicular distance from the rotating axis to *F*.

The torque τ relates the angular acceleration α and the moment of inertia *I*. This is analogous with the Newton's second law (*F* = *ma*), for which the angular quantities and the linear kinematics have the following connection.

Angular Quantity	**Linear Quantity**
α	a
τ	F
I	m

Example

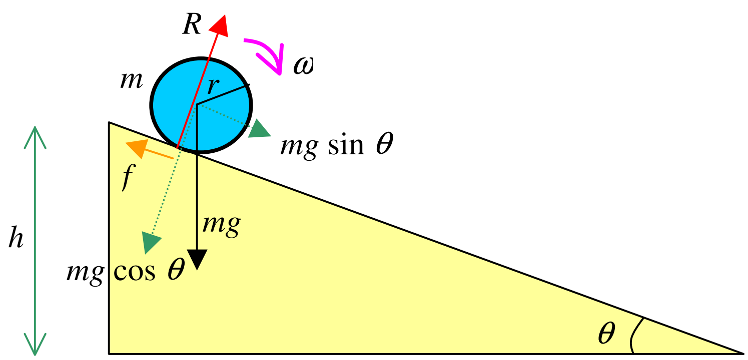
A disk is released from rest at the top of an inclined plane. If the disk rolls without slipping on the plane, calculate the following.

(a) The velocity of the disk when it reaches the lowest point of the inclined plane.

(b) The angular acceleration α and the linear acceleration a along the inclined plane.

(c) The time for the disk to roll down.

Answer:



By the conservation of energy, we have

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2,$$

where $v = r\omega$ and I is the moment of inertia of the disk. That is

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I(\frac{v}{r})^2$$

Now, we can calculate the velocity of the disk when it reaches the lowest point of the inclined plane.

$$v = \sqrt{\frac{mgh}{\frac{1}{2}m + \frac{1}{2}\frac{I}{r^2}}} \qquad (1)$$

We observe that the greater the moment of inertia, the smaller will be the velocity v. That's why a ring (with the same mass and radius) obtains a smaller velocity, when it rolls down and reaches the lowest point of the inclined plane.

Consider the forces along the inclined plane

$$mg\sin\theta - f = ma \qquad (2)$$

As the friction provides a torque, which rotates the disk, we can write

$$f\,r = I\alpha\,, \qquad (3)$$

where $a = r\,\alpha$ and α is the angular acceleration of the disk.

After solving (2) and (3), we obtain

$$\alpha = \frac{rmg\sin\theta}{I + mr^2} \qquad (4)$$

$$f = \frac{mg\sin\theta}{I + mr^2} I \qquad (5)$$

From (4), we observe that the greater the moment of inertia (N.B. for the same mass m and radius r but with different distribution of masses inside the rotating object), the smaller will be the angular acceleration of the rolling object when it rolls down the plane. The acceleration of the object is also smaller as $a = r\,\alpha$.

From (5), we can rewrite the equation as

$$f = \frac{mg\sin\theta}{I + mr^2}(I + mr^2 - mr^2) = mg\sin\theta - \frac{mg\sin\theta}{I + mr^2}(mr^2)\,.$$

Hence, the frictional force will be greater, if the rotating object has a greater moment of inertia.

Substitute the moment of inertia of the disk into (1), that is $I = \frac{1}{2}mr^2$, we have

$$v = \sqrt{\frac{4gh}{3}}\,,\ \alpha = \frac{2}{3}\frac{g\sin\theta}{r} \text{ and } a = r\alpha = \frac{2}{3}g\sin\theta\,.$$

The time for the disk to roll down is given by the relation $v = u + at$, where $u = 0$ (the disk is at rest when it starts to roll). Hence,

$$t = \frac{1}{\sin\theta}\sqrt{\frac{3h}{g}}\,.$$

Similar expressions can be obtained for a ring with the same mass and same radius. The moment of inertia of a ring is given by mr^2. Substituting it into the above equation, we have

$$v = \sqrt{gh}\,,\ \alpha = \frac{g\sin\theta}{2r} \ \text{and}\ a = r\alpha = \frac{g\sin\theta}{2}\,.$$

The time for a ring to roll down is given by the relation $v = u + at$, where u = 0 (the ring is at rest when it starts to roll). Hence,

$$t = \frac{2}{\sin\theta}\sqrt{\frac{h}{g}}\,.$$

Now, we can compare the time for disk and the ring to roll down the same distance. Of course, the time required for the ring to roll down the inclined plane of the same distance is longer.

7.5 Static equilibrium

Conditions for static equilibrium:

- The net force acting on the object must be zero,

$$\sum F_x = 0, \qquad \sum F_y = 0.$$

- The net torque acting on the object must be zero

$$\sum \tau = 0.$$

7.6 Rotational work

The work done of a force F for a small displacement Δx is given by

$$W = F\Delta x.$$

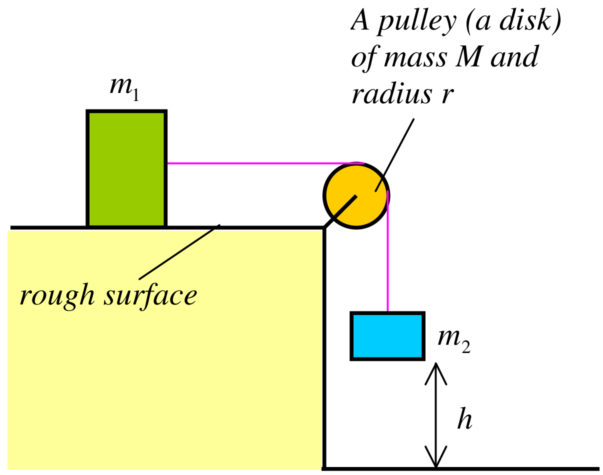
Since the small displacement Δx is obtained by a small angular displacement $\Delta\theta$, we have $\Delta x = r\,\Delta\theta$, hence

$$W = F\ (r\,\Delta\theta) = (F\,r)\,\Delta\theta = \tau\Delta\theta.$$

That is, the work done by a torque through a small angular displacement is given by

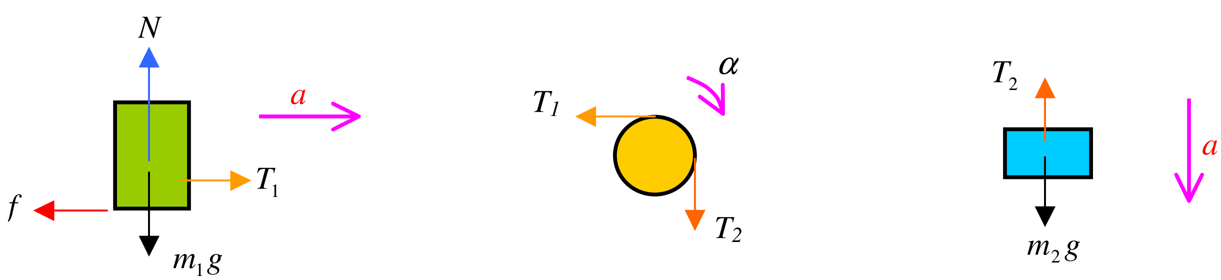
$$W = \tau \Delta\theta.$$

Example



The pulley is in the form of a disk and it rotates when m_2 starts to move down from rest. Given that the pulley has mass M and radius r, and there is no slipping between itself and the string during rotation. If the table surface is rough and the frictional coefficient is given as μ, find the accelerations of each block. Find the tensions of the string at the two sides of the pulley. Determine the velocity and the time required for m_2 to reach the ground.

Answer:



By using the Newton's second law, we obtain

$$\begin{cases} T_1 - f = m_1 a \\ T_2 r - T_1 r = I\alpha \\ m_2 g - T_2 = m_2 a \end{cases}$$

Since $f = \mu N$, $N = m_1 g$, $a = r\alpha$ and $I = \frac{1}{2}Mr^2$, we can rewrite the above set of equations as

$$\begin{cases} T_1 - \mu m_1 g = m_1 r\alpha \\ T_2 r - T_1 r = \dfrac{1}{2} M r^2 \alpha \\ m_2 g - T_2 = m_2 r\alpha \end{cases}$$

This is a system of equations with three equations and three unknowns. After solving, we obtain the angular acceleration of the pulley

$$\alpha = \frac{g}{r}\left(\frac{m_2 - \mu m_1}{\frac{1}{2}M + m_1 + m_2} \right).$$

The acceleration of blocks is given by $a = r\alpha = \left(\dfrac{m_2 - \mu m_1}{\frac{1}{2}M + m_1 + m_2} \right) g$.

The tension T_1 in the left hand side of pulley is

$$T_1 = m_1 g \left(\frac{\frac{1}{2}\mu M + m_2 + \mu m_2}{\frac{1}{2}M + m_1 + m_2} \right).$$

The tension T_2 in the right hand side of pulley is

$$T_2 = m_2 g \left(\frac{\frac{1}{2}M + m_1 + \mu m_1}{\frac{1}{2}M + m_1 + m_2} \right).$$

By the conservation of energy, we can write down

$$m_2 gh = \frac{1}{2} m_1 v^2 + \frac{1}{2} m_2 v^2 + \frac{1}{2} I\omega^2 + f\,h$$

The velocities of the two blocks are the same and $v = r\omega$, since the pulley has no sliding with the string. Substituting the moment of inertia of the pulley into the above equation and rearrange it, we obtain

$$\omega^2 = \frac{2gh}{r^2}\left(\frac{m_2 - \mu m_1}{\frac{1}{2}M + m_1 + m_2} \right).$$

Hence, the blocks move with the same velocity given by

$$v^2 = 2gh\left(\frac{m_2 - \mu m_1}{\frac{1}{2}M + m_1 + m_2}\right).$$

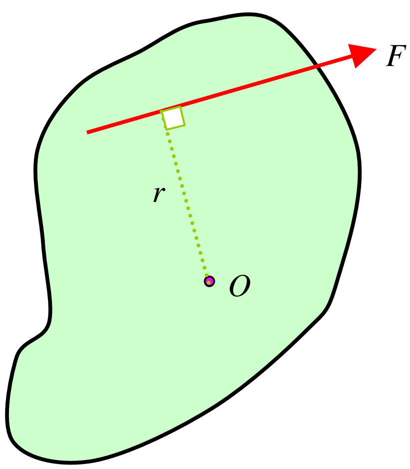
As the blocks move with acceleration a, the time required for m_2 to reach the ground is related by the equation $v = u + at$, where $u = 0$. That is

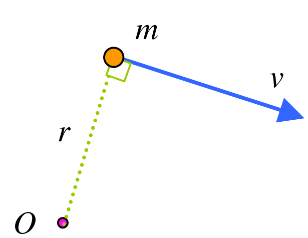
$$\sqrt{\frac{2gh(m_2 - \mu m_1)}{\frac{1}{2}M + m_1 + m_2}} = \left(\frac{m_2 - \mu m_1}{\frac{1}{2}M + m_1 + m_2}\right) g\,t.$$

Rearrange again, we obtain the time t as $\sqrt{\dfrac{\left(\frac{1}{2}M + m_1 + m_2\right)2h}{(m_2 - \mu m_1)g}}$.

7.7 Angular momentum

Recall that torque is the moment of force, e.g. $\tau = rF$, where r is the perpendicular distance of a point, say O, from F. Likewise, the angular momentum is defined as the moment of momentum, e.g. $L = r(mv) = rp$, where r is the perpendicular distance of a point of, say O, from v.





Note that the angular momentum can be rewritten as moment of inertia times the angular velocity.

$$L = r(mv) = r(mr\omega) = mr^2\omega = I\omega \qquad \text{Units: kgm}^2\text{s}^{-1}$$

Rotational Quantity	Linear Kinematics
Torque: $\tau = r\,F = I\alpha$	Force: $F = ma$
Angular momentum: $L = r\,p = I\,\omega$	Momentum: $p = mv$
Torque: Rate of change of angular momentum, $\tau = \dfrac{dL}{dt}$ (Recalled that $\alpha = \dfrac{d\omega}{dt}$)	Force: Rate of change of momentum, $F = \dfrac{dp}{dt}$ (Recalled that $a = \dfrac{dv}{dt}$)

Since $\tau = \dfrac{\Delta L}{\Delta t}$, we have

$$\Delta L = L_f - L_i = \tau \Delta t$$

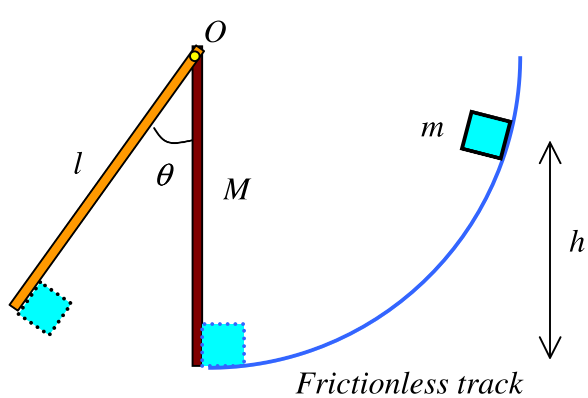
If $\tau = 0$, we have $\Delta L = 0$, that is $L_f = L_i$. This is the so-called conservation of angular momentum.

For linear kinematics, we have the conservation of momentum if there is no external force acting on the system, e.g. $\sum m_i v_i = \text{constant}$.

Similarly, we have the conservation of angular momentum if there is no external torque acting on the system, e.g. $\sum I_i \omega_i = \text{constant}$.

Example

A small block m collides with a uniform vertical rod after sliding down a frictionless track. If the block sticks together with the rod and the whole system rotates about point O, find the maximum height that the block can climb. You may neglect the dimension of the small block.

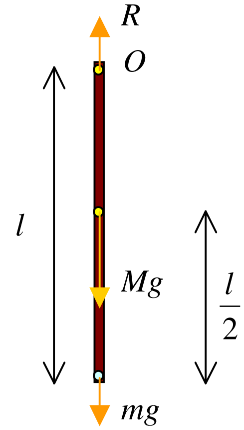


Answer:

The velocity of block m just before the collision can be found out by using the principle of conservation of energy

$$\frac{1}{2}mv^2 = mgh$$

i.e. $\quad v = \sqrt{2gh}$



For the Collision:

As all external forces (R, Mg, mg) pass through point O, and they give no external torque to the system, that is

$$\sum \tau_{ext} = 0.$$

Now, we apply the principle of conservation of angular momentum to discuss the collision.

Before collision, the angular momentum about O is given by

$$0 + l(mv) = l\,mv.$$

After collision, the angular momentum about O is given by

$$l(mV) + I\omega = l\,mV + (\frac{1}{3}M\,l^2)\,\omega,$$

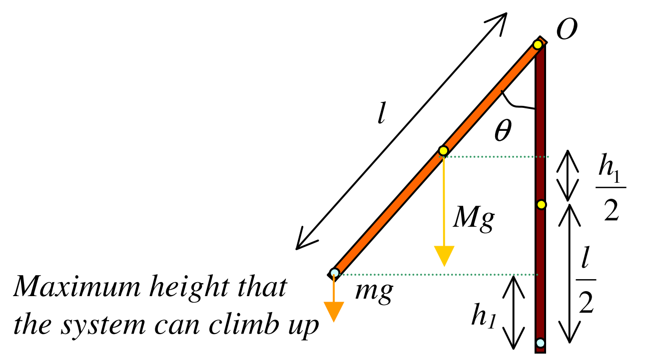
where V is the velocity of the combined system just after collision, $V = l\omega$, and I is the moment of inertia of the rod about O. Since the angular momentum before and after the collision are conserved, we have

$$l\,mv = l\,mV + \frac{1}{3}M\,l^2\omega.$$

After plugging in $V = l\omega$, one obtains

$$mvl = l^2 m\omega + \frac{1}{3}M\,l^2\omega$$

which gives $\omega = \dfrac{mv}{l(m + \frac{1}{3}M)} = \dfrac{m\sqrt{2gh}}{l(m + \frac{1}{3}M)}$



Just after collision

and $V = l\omega = \dfrac{m\sqrt{2gh}}{m + \frac{1}{3}M}$.

By the conservation of energy, one can write: ***Total loss in K.E. = Total Gain in P.E.***

That is

$$\frac{1}{2}mV^2 + \frac{1}{2}I\omega^2 = mgh_1 + Mg\frac{h_1}{2} \ ,$$

where h_1 is the maximum height that the block can climb up after collision and ω is obtained before. After substituting expressions for I, V and ω into the above equation, we obtain

$$h_1 = \frac{m^2 h}{(m + \frac{1}{3}M)(m + \frac{1}{2}M)} .$$