Hong Kong Physics Olympiad Lesson 8

8.1 Periodic motion

A motion that repeats itself over and over is referred to as periodic motion.

Some useful quantities:

- Period, T

  T = Time required for one cycle of a periodic motion   Unit: second

- Frequency, f

  f = The number of oscillation per second   Unit: cycle s^{-1} = s^{-1} or Hz

  Note that $T = 1/f$.

- Angular velocity, ω

  ω = Angular displacement per unit time   Unit: radian s^{-1} = s^{-1}

  Note that 1 period → 2π, hence

  $\omega = 2\pi f$ (In words, the angular displacement is $2\pi f$ radian per second)

  or $\omega = \dfrac{2\pi}{T}$

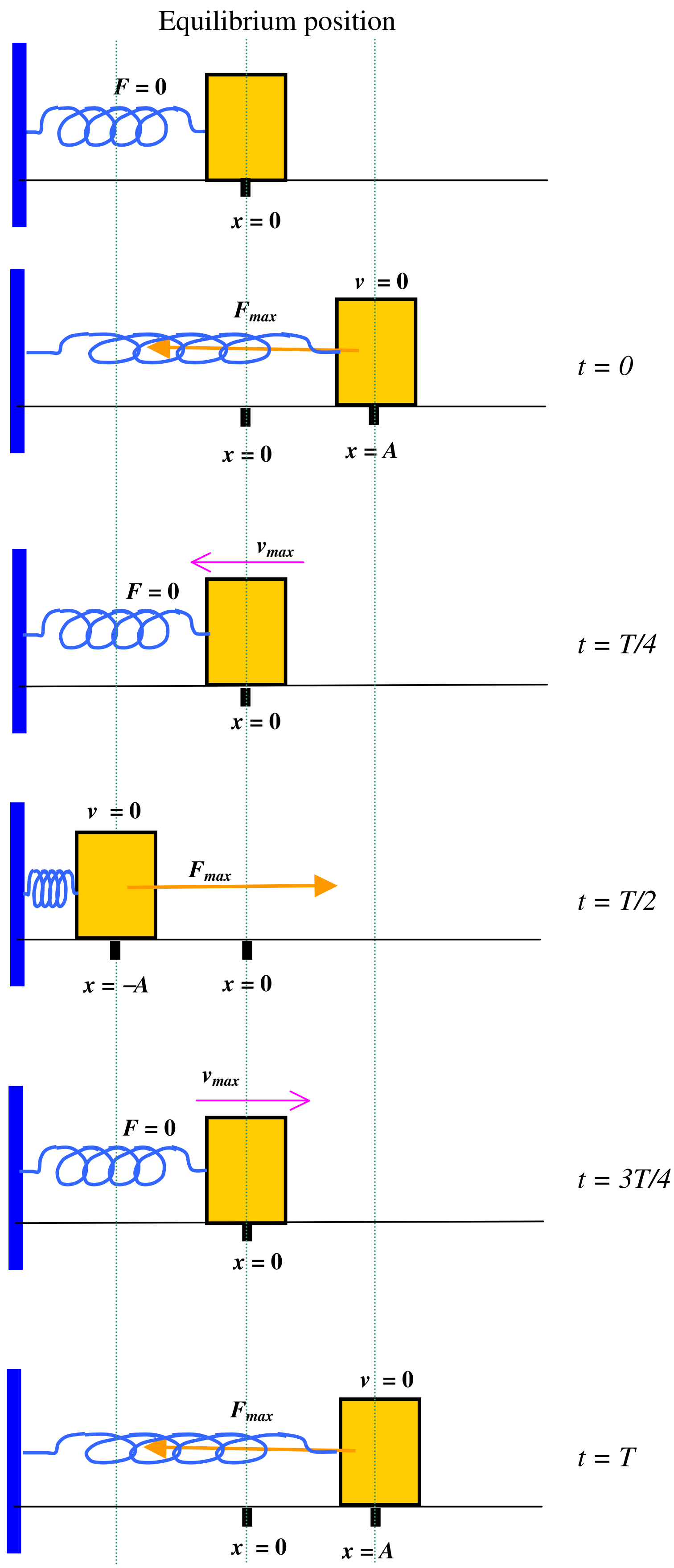
- Angular displacement, θ

  θ = Angular displacement in time t

  $\theta = \omega t$

- Amplitude = The maximum displacement (angular displacement) of the motion.

8.2 Simple harmonic motion

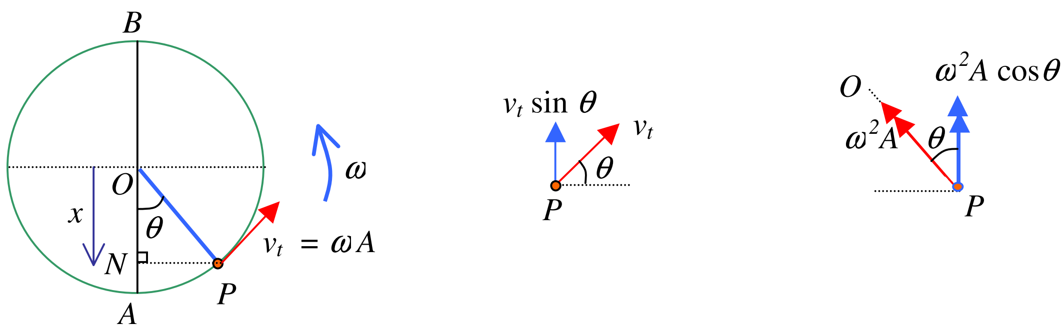


Hooke's law states that the restoring force F is proportional to the displacement from its equilibrium position:

$$F = -kx.$$

The negative sign is appeared in the force equation, which shows that the restoring force and the displacement vector from the equilibrium position are in opposite direction. The equilibrium position is defined as the point where the net force acting on the vibrating mass is zero. For the above example, $x = 0$ is the equilibrium point.

8.3 Position, velocity and acceleration in S.H.M.



Let the projection velocity of the particle on AB as v which is a component of the tangential velocity v_t of the particle. We label the radius of the circle, that is distance OA as A.

$$x = A\cos(\omega t) \qquad (1)$$

$$v = -A\omega\sin(\omega t) \qquad (2)$$

$$a = -A\omega^2\cos(\omega t) \qquad (3)$$

Hence, we have $v^2 = \omega^2(A^2 - x^2)$ and $a = -\omega^2 x$. Note that the latter equation is a second order differential equation in the form

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad \text{or} \quad \ddot{x} + \omega^2 x = 0.$$

Note also that

- $x_{max} = A$

- $v_{\max} = A\omega$ and $a_{\max} = A\omega^2$.
- From (1) and (2), we know that x and v differs by $\pi/2$.
- From (1) and (3), we know that x and a are out of phase, that is if x is at its maximum, then a must be at its minimum and vice versa.

For the motion of the spring-mass system mentioned above

$$F = ma = -kx,$$

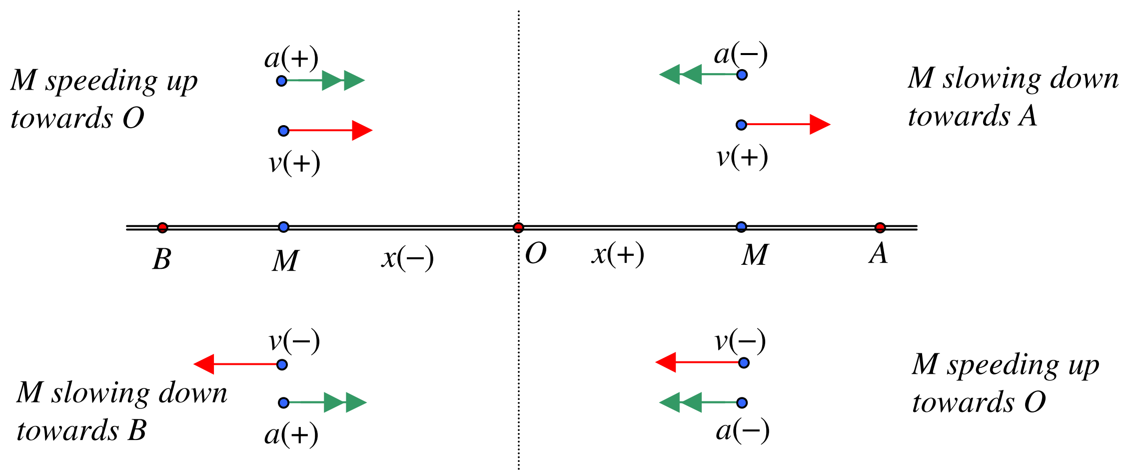
which implies $a = -\omega^2 x$, where $\omega = \sqrt{\frac{k}{m}}$ is the natural frequency of the system.

The period of the system T is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}} = 2\pi\sqrt{\frac{m}{k}}.$$

Remark:

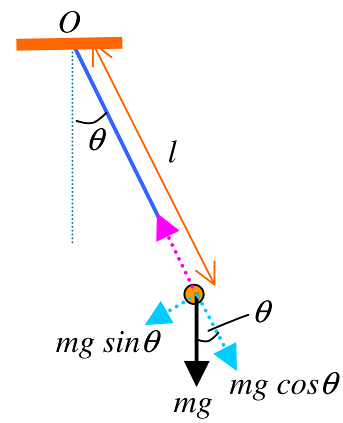
If a particle M is oscillating about the equilibrium position O and points A and B are the extremes of the motion, then we have the following signs for the dynamic quantities.



8.4 Simple pendulum

We observe that the only force which points along the tangential direction is $-mg\ sin\theta$. The negative sign indicates that the force is towards O while the displacement is measured along the arc from O in the opposite direction.

$$-mg\ sin\ \theta = ma$$



For small oscillation, θ is small, $sin\ \theta \sim \theta$, but $x = l\theta$, hence we have

$$\theta = x/l \qquad \text{and} \qquad -mg\frac{x}{l} = ma$$

Now, $a = -\frac{g}{l}x = -\omega^2 x$, where $\omega = \sqrt{\frac{g}{l}}$ is the natural frequency of the system.

The period of oscillation T can be obtained by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{g/l}} = 2\pi\sqrt{\frac{l}{g}}\,.$$

The whole calculation can be obtained by considering the torque of the system about O

$$-l\ mg\ sin\ \theta = ml^2\alpha,$$

where ml^2 is the moment of inertia of the pendulum.

Using the same arguments, we have $-l\ mg\ \theta = ml^2\alpha$, rearrange the equation, we obtain

$$\alpha = -\frac{g}{l}\theta = -\omega^2\theta \qquad \text{where } \omega = \sqrt{\frac{g}{l}}\,.$$

Example

A particle moving with S.H.M. has velocities of 4 cm/s and 3 cm/s at distances of 3 cm and 4 cm respectively from the equilibrium position. Find

(a) the amplitude of the oscillation,

(b) the period,

(c) the velocity of the particle as it passes through the equilibrium position.

Answer:

(a) We know that $v^2 = \omega^2(A^2 - x^2)$, we can write down two equations by the given information and then solve for the two unknowns, A and ω respectively.

$$4^2 = \omega^2(A^2 - 3^2) \qquad (1)$$

$$3^2 = \omega^2(A^2 - 4^2) \qquad (2)$$

Dividing (1) by (2) and then we have $\dfrac{16}{9} = \dfrac{A^2 - 9}{A^2 - 16}$,

which gives $A = \pm 5$.

Since A is the amplitude, it must be positive, we have $A = 5$.

(b) Obviously, we obtain, $\omega = 1\text{s}^{-1}$.

Since $T = \dfrac{2\pi}{\omega} = 2\pi\, s$

(c) Plug in $A = 5$ and $x = 0$ into the equation $v^2 = \omega^2(A^2 - x^2)$, we obtain the velocity of the particle when it passes through the equilibrium position.

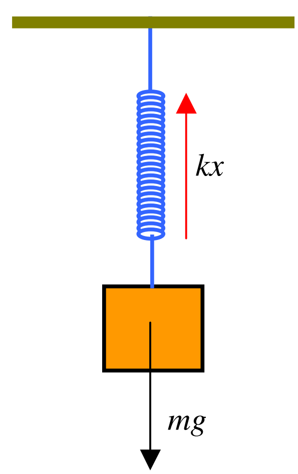
$$v^2 = 1^2(5^2 - 0^2)$$

$$v = \pm 5\, cms^{-1}$$

Example

A light spring is loaded with a mass under gravity. If the spring extends by 10 cm, calculate the period of small vertical oscillation.

Answer:



Equilibrium

For equilibrium, the downward force, that is, the weight equals the restoring force of the spring, $mg = kx$.

Now we have $k = \dfrac{mg}{x}$.

Since the period of the oscillation is given by $T = 2\pi\sqrt{\dfrac{m}{k}}$, hence we have

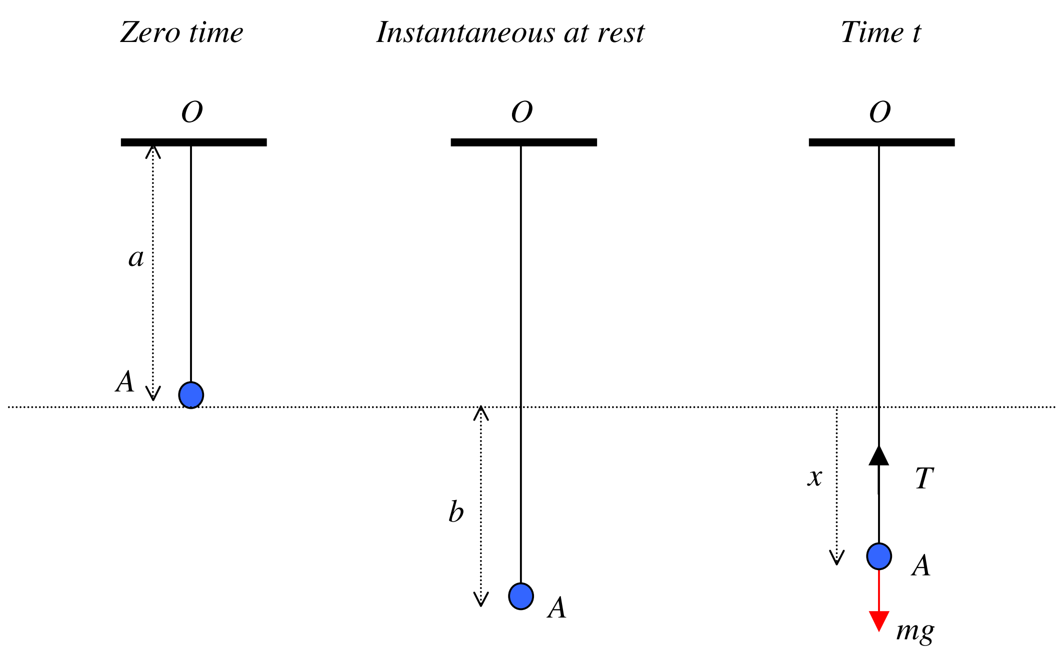
$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{x}{g}} = 2\pi\sqrt{\frac{0.1}{9.8}} = 0.63\, s.$$

Note that the period of the mass-spring system are the same, no matter what the orientation of the system is, horizontal, vertical or inclined.

Example

A particle of mass m is attached to one end A of an elastic string of modulus λ and natural length a. The other end of the string is fixed to a point O. The system is released from rest when A is vertically below O and the length OA is a. Show that the particle reaches its next position of instantaneous rest when its depth below O is $a\left(1+\frac{2mg}{\lambda}\right)$. Show also that the particle moves with simple harmonic motion about a point at a distance $a\left(1+\frac{mg}{\lambda}\right)$ below O.

Answer:



We consider the conservation of energy, $mgb = \frac{1}{2}\left(\frac{\lambda}{a}\right)b^2$, where b is the extension extra from its natural length. Note that $\frac{\lambda}{a}$ has the same meaning as the force constant k in a spring. Now, we have $b = \frac{2mga}{\lambda}$.

The particle reaches its next position of instantaneous rest when its depth below O is $a + \frac{2mga}{\lambda} = a\left(1 + \frac{2mg}{\lambda}\right)$.

The equation of motion is $mg - T = m\ddot{x}$, where $T = \frac{\lambda}{a}x$ and $\ddot{x}$ is the acceleration of the particle.

The equation $mg - \frac{\lambda}{a}x = m\ddot{x}$ gives $\ddot{x} = -\frac{\lambda}{ma}x + g$.

Rearrange the equation again, we have $\ddot{x} = -\frac{\lambda}{ma}\left(x - \frac{ma}{\lambda}g\right)$.

Making the substitution $z = x - \frac{ma}{\lambda}g$, we have $\ddot{z} = \ddot{x}$ and

$$\ddot{z} = -\frac{\lambda}{ma}z.$$

This is an equation for simple harmonic motion with $\omega^2 = \frac{\lambda}{ma}$. The period is given by $\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{ma}{\lambda}}$. As $z = 0$ is the new equilibrium position, $z = x - \frac{ma}{\lambda}g = 0$ gives $x = \frac{ma}{\lambda}g$. The particle moves about the point $a + \frac{ma}{\lambda}g = a\left(1 + \frac{m}{\lambda}g\right)$ below O.

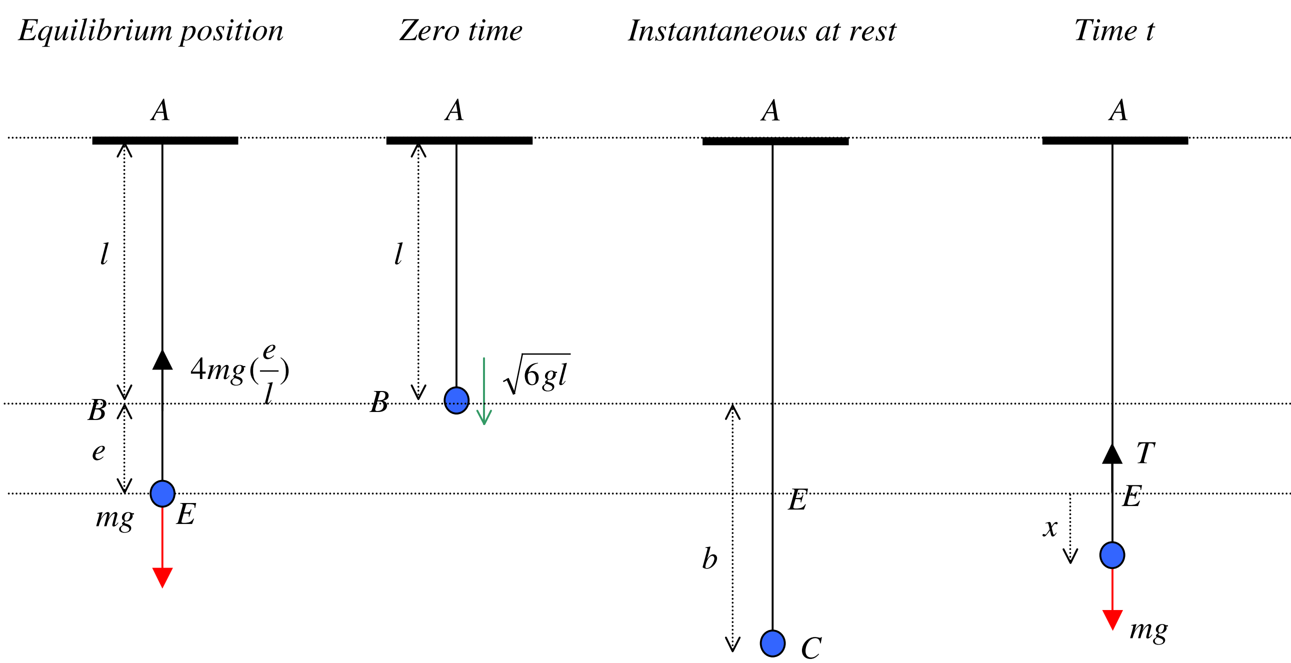
Remarks:

Restoring force in an elastic string = $\lambda\left(\frac{b}{a}\right)$, where a is the natural length of string and b is the extension. The modulus λ is the proportional constant which relates the restoring force and the fractional extension.

Example

One end of a light elastic string, of natural length l and modulus of elasticity $4mg$, is fixed to a point A and a particle of mass m is fastened to the other end, The particle hangs in equilibrium vertically below A. Find the extension of the string. The particle is now held at the point B at a distance l vertically below A and projected vertically downwards with speed $6gl$. If C is the lowest point reached by the particle, prove that the motion from B to C is simple harmonic of amplitude $\frac{5}{4}l$. Prove also that the time taken by the particle to move from B to C is $\frac{1}{2}\left[\frac{1}{2}\pi + \sin^{-1}\left(\frac{1}{5}\right)\right]\sqrt{\frac{l}{g}}$.

Answer:



At equilibrium, we have $mg = 4mg\left(\frac{e}{l}\right)$, where e is the extension from the string's natural length. Hence, we obtain $e = \frac{l}{4}$.

By the conservation of energy, we can write $\frac{1}{2}m(6gl) + mgb = \frac{1}{2}(4mg)\left(\frac{b^2}{l}\right)$.

Simplifying the above expression, we have $2b^2 - bl - 3l^2 = 0$ which gives $b = \dfrac{3l}{2}$.

The equation of motion is $mg - T = m\ddot{x}$, where $T = 4mg\left(\dfrac{x + \frac{l}{4}}{l}\right)$.

Now, we obtain the S.H.M. equation $\ddot{x} = -\dfrac{4g}{l}x$. The motion from B to C is S.H.M. with amplitude $b - e = \dfrac{5l}{4}$ and $\omega = \sqrt{\dfrac{4g}{l}}$.

The time taken to move from B to E is t_1, where $\dfrac{l}{4} = \dfrac{5l}{4}\sin\left(\sqrt{\dfrac{4g}{l}}t_1\right)$. Hence, we obtain

$$t_1 = \frac{1}{2}\sqrt{\frac{l}{g}}\sin^{-1}\left(\frac{1}{5}\right).$$

Remark: The above method is a short cut without talking about the phase of oscillation. In fact the displacement-time relation should be written as $x = \dfrac{5l}{4}\sin(\omega t + \delta)$ which gives $-\dfrac{l}{4} = \dfrac{5l}{4}\sin(\omega(0) + \delta)$ when $t = 0$. Thus, we obtain $\delta = -\sin^{-1}\left(\dfrac{1}{5}\right)$. At $t = t_1$, we have $0 = \dfrac{5l}{4}\sin(\omega(t_1) + \delta)$ which gives $0 = \omega t_1 + \delta$. Plugging in ω and the phase δ, we obtain $t_1 = \dfrac{1}{2}\sqrt{\dfrac{l}{g}}\sin^{-1}\left(\dfrac{1}{5}\right)$.

The time taken to move from E to C is $\dfrac{1}{4}$ period, i.e. $\dfrac{1}{4}\left(\dfrac{2\pi}{\omega}\right) = \dfrac{1}{4}\left(2\pi\sqrt{\dfrac{l}{4g}}\right) = \dfrac{\pi}{4}\left(\sqrt{\dfrac{l}{g}}\right)$.

The time taken to move from B to C is $\dfrac{1}{2}\sqrt{\dfrac{l}{g}}\sin^{-1}\left(\dfrac{1}{5}\right) + \dfrac{\pi}{4}\left(\sqrt{\dfrac{l}{g}}\right)$, that is $\dfrac{1}{2}\left[\dfrac{1}{2}\pi + \sin^{-1}\left(\dfrac{1}{5}\right)\right]\sqrt{\dfrac{l}{g}}$.

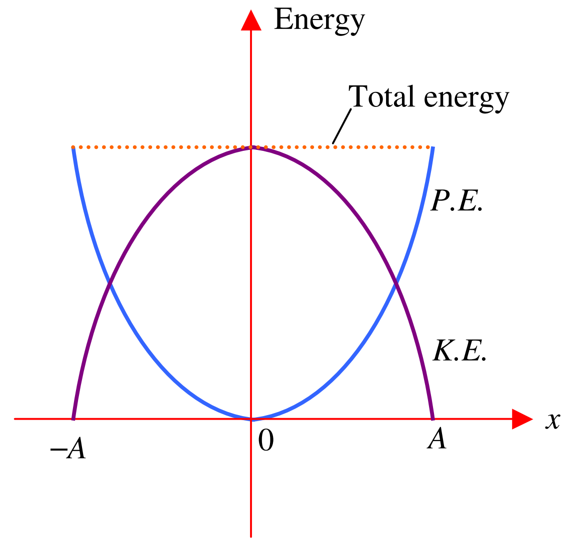
8.5 Energy of simple harmonic motion

If the oscillation involves no dissipation in energy, e.g. friction, the summation of the kinetic energy and the potential energy becomes conserved, that is, the total energy is a constant. In the spring-mass system, we have the total energy E

$$E = K.E + P.E. = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}$$

Plug in $x = A$ and $v = 0$, the constant E is obtained as

$$\frac{1}{2}kx^2 \text{ which equals } \frac{1}{2}m\omega^2 A^2.$$



When we plot the graph $K.E.$ against x, we find that it is a quadratic curve.

$$K.E = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}m\omega^2 A^2 - \frac{1}{2}m\omega^2 x^2,$$

Recalled that the first term $\frac{1}{2}m\omega^2 A^2$ is a constant.

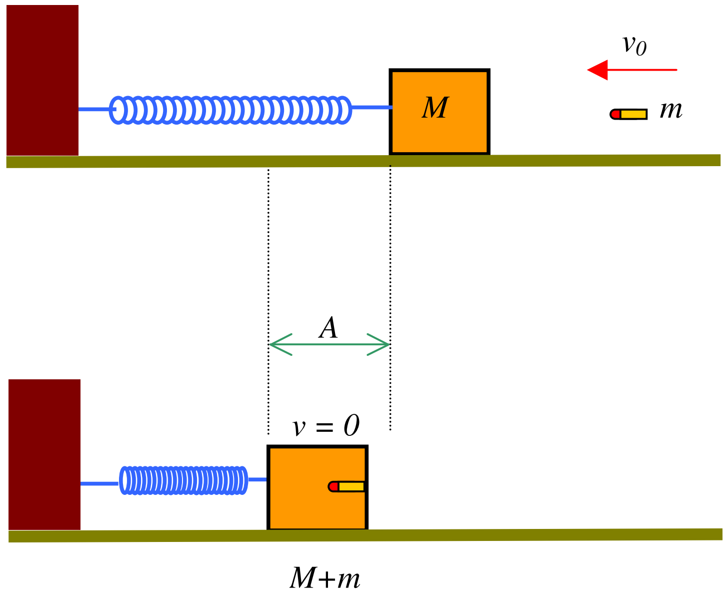
Note also that the graph $P.E.$ against x is also a quadratic curve

$$P.E. = \frac{1}{2}kx^2.$$

Example

A bullet of mass m embeds itself in a block of mass M, which is attached to a spring of force constant k, just after collision. If the initial speed of the bullet is v_0, find (a) the maximum compression of the spring and (b) the time for the bullet-block system to come to rest.

Answer:



By the conservation of momentum, we have

$$mv_0 = (m+M)v$$

That is the combined velocity $v = \dfrac{mv_0}{m+M}$.

The kinetic energy of the system at that moment is $\frac{1}{2}(m+M)v^2$, which changes to elastic potential energy when the system comes to rest

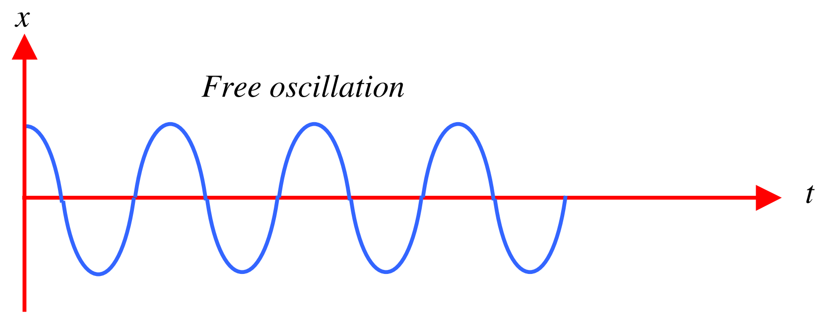
$$\frac{1}{2}(m+M)v^2 = \frac{1}{2}kA^2,$$

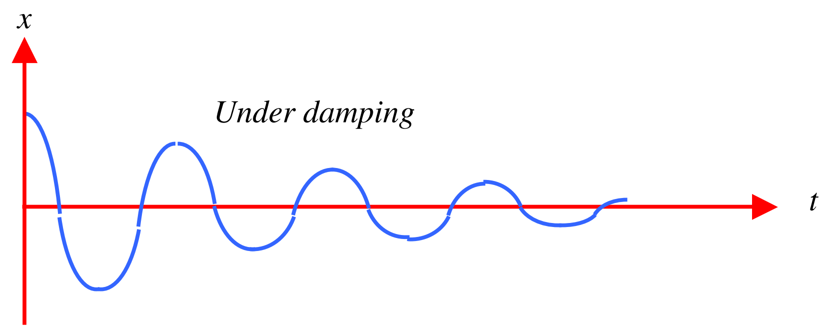
where A is the maximum distance that the block and the bullet can move.

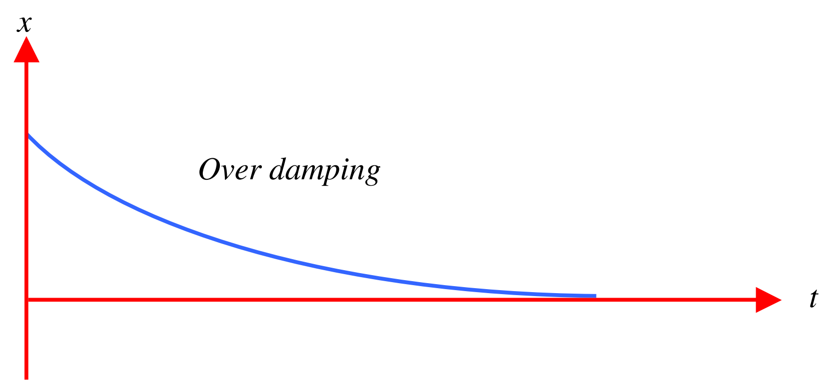
Substituting the expression for v, we obtain $\dfrac{m^2 v_0^2}{m+M} = kA^2$ and rearrange it, we have the amplitude A given by $\dfrac{mv_0}{\sqrt{k(m+M)}}$.

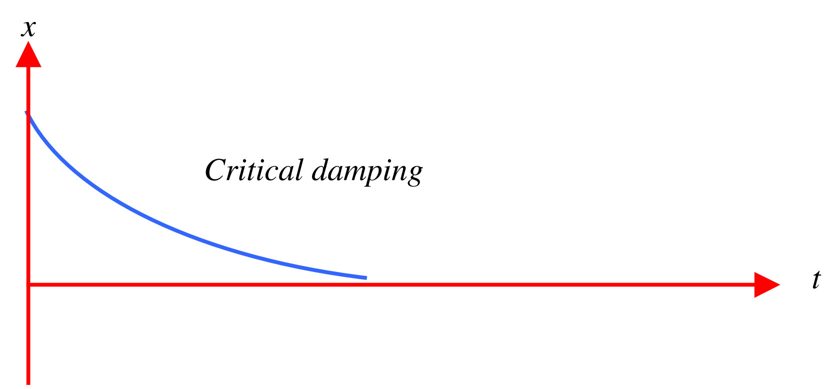
Since the system is a spring-block oscillating system, the period T is given by $T = 2\pi\sqrt{\dfrac{m+M}{k}}$, hence, the time for the travels from just impact to instantaneous at rest is given by $T/4$.

8.6 Damped and forced oscillations









There are external forces which act on the system other than the restoring force. If the external force damps the system and dissipates the system's energy, it is called the damping force. Example of this is the friction or the viscous force. The motion is then a damped oscillation. If energy is pumped into the system by an external force, the force is called by driving force and the oscillation is named as forced oscillation.

When the driven force has its frequency equals the natural frequency of the system, the system is under resonance.