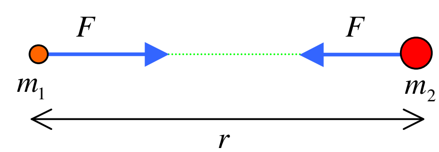
Hong Kong Physics Olympiad Lesson 9

9.1 Gravity

Newton's law of universal gravitation states that the gravitational force F between any two bodies of mass m_1 and m_2 separated by a distance r, is described by

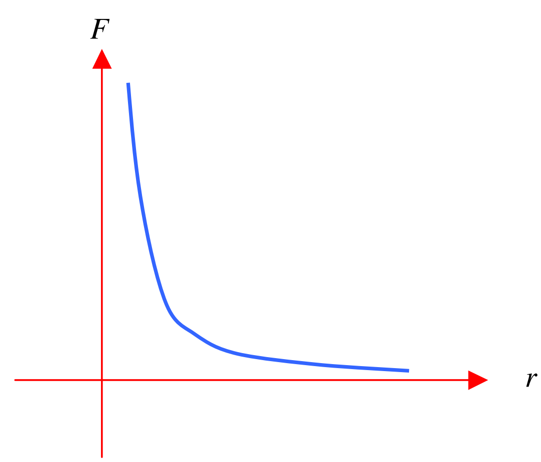
$$F \propto \frac{m_1 m_2}{r^2}.$$



It is a center-to-center attraction between all forms of matter. The force of gravity between any two bodies varies directly in proportion to the product of their masses and inversely with their separation squared.

The proportional constant is referred to as the universal gravitation constant G. Its value is $G = 6.67\times10^{-11}$ Nm^{2}kg^{-2}. Now we can write

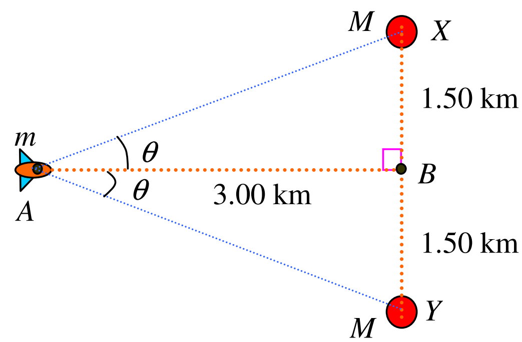
$$F = G\frac{m_1 m_2}{r^2}$$



Example

Twin asteroids X and Y, having the same mass ($M = 3.5\times10^{18}$ kg) are located 3.00 km apart. Find the net gravitational force of the spaceship when it is located at positions A and B, as shown in figure. Given that the mass of the spaceship is $m = 2.50\times10^7$ kg.

Answer:



When the spaceship is at A:

The distance $AX = AY = (3000^2+1500^2)^{1/2} = 3350$ m.

The angle $\theta = \tan^{-1}(\frac{1500}{3000}) = 26.6^\circ$.

The attractive force between the spaceship and asteroid X is $F = G\frac{mM}{AX^2}$ along AX.

The attractive force between the spaceship and asteroid Y is $F = G\frac{mM}{AY^2}$ along AY.

But the vertical components of the two attractive forces are of opposite direction and equal in magnitudes, they counterbalance each other. Hence the net force is summation of the two horizontal forces.

$$F_{net} = G\frac{mM}{AX^2}\cos\theta + G\frac{mM}{AY^2}\cos\theta$$

$$F_{net} = 2G\frac{mM}{AX^2}\cos\theta$$

$$F_{net} = 2(6.67\times10^{-11})\left(\frac{(2.50\times10^7)(3.50\times10^{18})}{3350^2}\right)\cos 26.6^\circ = 9.30\times10^8\ N$$

When the spaceship is at B:

The attractive force between the spaceship and asteroid X is $F = G\frac{mM}{BX^2}$ along BX.

The attractive force between the spaceship and asteroid Y is $F = G\frac{mM}{BY^2}$ along BY.

Note that they are of the same magnitude but opposite in direction, hence the net force acting on the spaceship is zero.

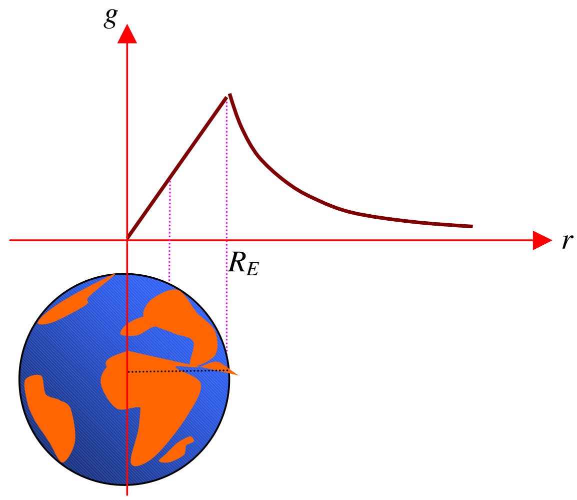
9.2 Gravitational field

Gravitational field is defined as the gravitational force per unit mass at a point, in terms of a test mass, due to the earth. It is a directional interaction between particles. And, the point of view is called action-at-a-distance.

$$g = \frac{F}{m} = \frac{G\frac{mM_E}{r^2}}{m} = G\frac{M_E}{r^2},$$

where r is the distance of the test mass from the center of earth ($r > R_E$).

The unit of gravitational field is the same as that of acceleration, i.e. N/kg, or ms^{-2}.



When $r < R_E$, the gravitational field g is proportional to the distance of the test mass and the center of earth. When $r \sim R_E$, the gravitational field g is approximated by a common value 9.81 ms^{-2}. It is quite often to use this value in our calculations, as a few meters or even few kilometers alter the value of g not much. When $r > R_E$, the curve is an inverse square law.

Remark:

Shell Theorem #1:

A uniformly dense spherical shell attracts an external particle as if all the masses of the shell were concentrated at its center.

Shell Theorem #2:

A uniform dense spherical shell exerts no gravitational force on a particle located anywhere inside it.

When $r < R_E$, we observe a linearity in the graph and it can be explained as follows. By the Shell theorem, the masses outside the test mass give no force to the test mass, the gravitational force is solely resulted from the remaining mass ($M = \frac{4}{3}\pi r^3 \rho$). However, the mass M can be regarded as a point mass which locates at the center of earth, hence can express the gravitational field as

$$g = \frac{GM}{r^2} = \frac{G(\frac{4}{3}\pi r^3 \rho)}{r^2} = \frac{4}{3}G\pi r\rho = \frac{4}{3}G\pi r(\frac{M_E}{\frac{4}{3}\pi R_E^3}) = \frac{GM_E}{R_E^3} r .$$

Example

If g' is the acceleration due to gravity at a distance a from the center of the earth, where $a > R_E$, R_E being the radius of the earth. Show that g' relates to g by

$$g' = \frac{R_E^2}{a^2} g$$

where g is the acceleration due to gravity at the surface of the earth. Show further that if a is located a distance h higher than the surface of the earth, then

$$g' = (1 - \frac{2h}{R_E}) g .$$

Answer:

Since $g' = \frac{GM}{a^2}$ and $g = \frac{GM}{R_E^2}$, we obtain

$$g' = \frac{R_E^2}{a^2} g$$

As $a = R_E + h$, we have $g' = \frac{R_E^2}{a^2} g = \frac{R_E^2}{(R_E + h)^2} g = \frac{1}{(1 + \frac{h}{R_E})^2} g = (1 + \frac{h}{R_E})^{-2} g$.

If h is very small compared to R_E, we can neglect the higher order terms after expanding the series. Hence we can write

$$g' = (1 - \frac{2h}{R_E}) g$$

When h is extremely small, h/R_E approaches zero. That's why we quite often approximate g' by g.

Example

Suppose a tunnel could be dug through the earth from one side to the other along a diameter. A particle of mass m is dropped into the tunnel from rest at the surface.

(a) What is the force on the particle when it is a distance r from the center?

(b) What is the period of the particle in the tunnel?

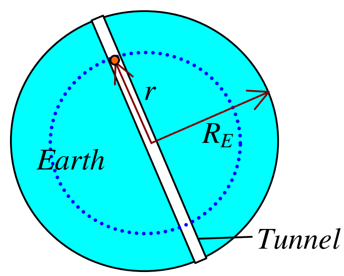
(c) What is the speed of the particle when it is a distance r from the center?

(d) Evaluate the speed at $r = 0$.

Neglect all frictional forces and assume that the earth has a uniform density.

Answer:

(a) According to the shell theorem #2, we conclude that the gravitational force on the particle is due to that portion of the earth lies inside the sphere of radius r. From shell theorem #1, we know that the sphere enclosed by the radius r can be regarded as a particle of the same mass of the sphere.



The force on the particle $F = mg = \frac{GmM_E}{R_E^3} r$. ($g$ is obtained in the previous section)

(b) In vectorial description $\vec{F} = -\frac{GmM_E}{R_E^3}\vec{r}$, that is $k = \frac{GmM_E}{R_E^3}$. Hence, the particle moves in S.H.M with $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{GM_E}{R_E^3}}$. The particle moves with period T

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_E^3}{GM_E}} = 2\pi\sqrt{\frac{(6.37\times10^6 m)^3}{(6.67\times10^{-11} Nm^2/kg^2)(5.98\times10^{24} kg)}} = 84.3 \text{ minutes.}$$

(c) By the conservation of energy: $K_S + U_S = K_r + U_r$

We have $0 + \frac{1}{2}kR_E^2 = \frac{1}{2}mv^2 + \frac{1}{2}kr^2$.

After solving, $v = \sqrt{\frac{k}{m}(R_E^2 - r^2)} = \sqrt{\frac{GM_E}{R_E^3}(R_E^2 - r^2)}$.

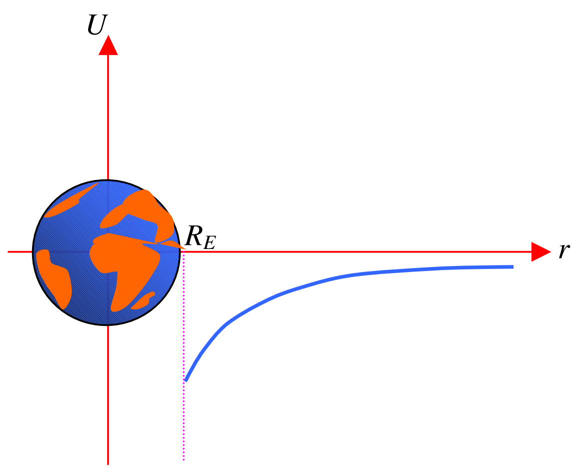
(d) Plug in $r = 0$, we obtain the velocity of the particle at the center of earth

$$v = \sqrt{\frac{GM_E}{R_E}} = \sqrt{\frac{(6.67\times10^{-11}\,Nm^2/kg^2)(5.98\times10^{24}\,kg)}{6.37\times10^6\,m}} = 7.91\times10^3\,m/s.$$

9.3 Gravitational potential energy

The gravitational potential energy of a system consisting of a mass m and a distance r from the center of the earth is

$$U = -G\frac{M_E m}{r} \qquad \text{Unit: joule, J}$$



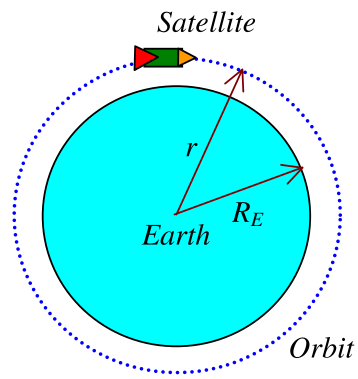
where M_E is the mass of the earth. The negative sign represents the attractive nature of the gravitational force. And, U represents the work done of the attractive force to move a mass m from a point r measured from the center of the earth to infinity. Note that U approaches zero as r approaches infinity.

9.4 Satellite orbits

The centripetal force for the satellite to run its orbit can be provided by the gravitational attraction.

$$\frac{mv^2}{r} = \frac{GM_E m}{r^2}$$

$$v^2 = \frac{GM_E}{r} = \frac{GM_E}{R_E^2}\frac{R_E^2}{r} = g\frac{R_E^2}{r}$$



Given that the earth's radius is 6400 km and the orbit of the satellite is 200 km from the earth's surface, then

$$r = (6400 + 200)\text{ km} = 6600\text{ km} = 6.6\times10^6\text{ m}$$

$$v^2 = 9.81(\frac{R_E^2}{r}) = 9.81\times\frac{(6.4\times10^6)^2}{6.6\times10^6}$$

$$v \approx 8\,kms^{-1}$$

The time for the satellite to make one complete orbit of the earth is

$$T = \frac{2\pi r}{v}$$

$$T \approx 86\,\text{minutes}$$

Example

A satellite is moving in a circular orbit of radius R and has a period of 24 hours. What is the period of a satellite which moves in a circular orbit of radius $R/4$?

Answer:

The centripetal force of the satellite is given by the gravitational attraction. Hence, we can write

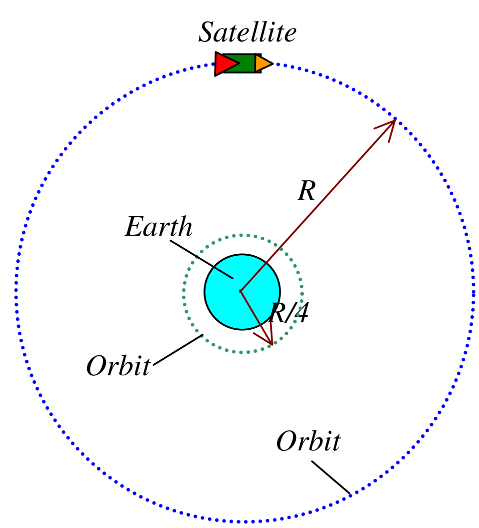
$$m\omega^2 R = \frac{GMm}{R^2},$$

where m and M are the mass of satellite and the earth respectively. As $T = \frac{2\pi}{\omega}$, the above equation can be rewritten and expressed in terms of the period T, e.g.

$$m(\frac{2\pi}{T})^2 R = \frac{GMm}{R^2}. \qquad (1)$$

When the radius of orbit is $R/4$, we have

$$m(\frac{2\pi}{T'})^2 (\frac{R}{4}) = \frac{GMm}{(\frac{R}{4})^2}, \qquad (2)$$

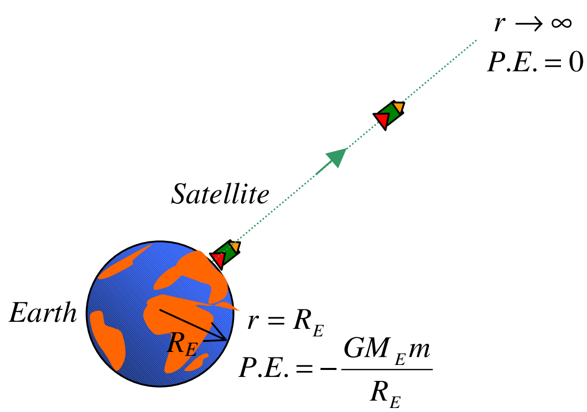


where T' is the new period.

Divide equation (1) by equation (2), we have $T' = \frac{T}{8}$. Plugging in T = 24 hours, we obtain T' = 3 hours.

9.5 Launching a satellite

The satellite can escape from the earth, whenever it has enough energy. Note that the potential energy of the satellite is zero when it escapes from the earth.



$$\frac{1}{2}mv^2 - \frac{GM_E m}{R_E} = 0$$

$$v = \sqrt{\frac{2GM_E}{R_E}} = \sqrt{2(\frac{GM_E}{R_E^2})R_E} = \sqrt{2gR_E}$$

Plugging in the radius of earth $R_E = 6.4 \times 10^6$ m and the gravitational acceleration at the earth's surface g, we obtain the escape velocity v = 11 kms^{-1}.

9.6 Mechanical energy of a satellite

Consider a satellite moving along a circular orbit with tangential velocity v at a distance r from the center of earth. The potential energy of the satellite is

$$U(r) = -\frac{GM_E m}{r}.$$

The total mechanical energy = $K.E.$ + $P.E.$ = $K.E.$ + $U(r)$

Hence we have $E = \frac{1}{2}mv^2 - \frac{GM_E m}{r}$.

But the centripetal force is provided by the gravitational attraction

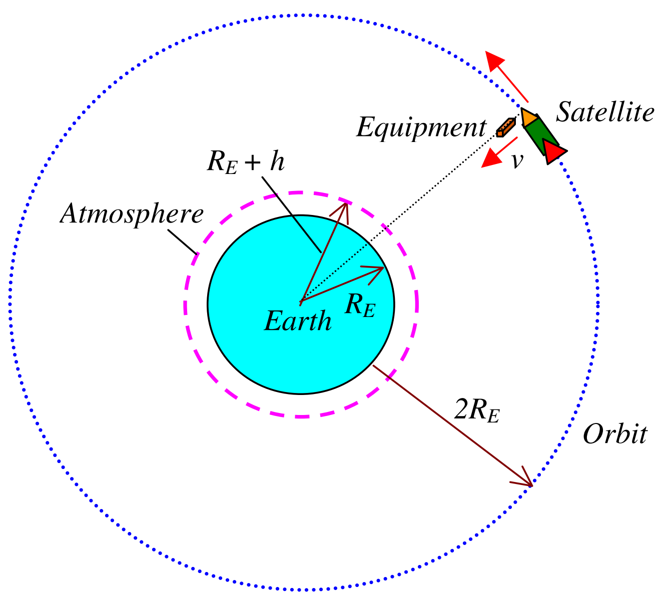
$$\frac{mv^2}{r} = \frac{GM_E m}{r^2}$$

which gives the velocity v, i.e. $v^2 = \frac{GM_E}{r}$. Finally, we obtain the total mechanical energy

$$E = \frac{1}{2}m(\frac{GM_E}{r}) - \frac{GM_E m}{r} = -\frac{GM_E m}{2r}$$

Example

A satellite, orbiting at an altitude of two earth's radii above its surface, launches an equipment of mass m toward the earth's center with a speed of v = 525 m/s. With what speed v_f does the equipment enter the earth's atmosphere (a distance of h = 100 km above it's surface)?



Answer:

By conservation of energy, we have $K_i + U_i = K_f + U_f$. That is

$$\frac{1}{2}mv_i^2 - \frac{GM_E m}{r_i} = \frac{1}{2}mv_f^2 - \frac{GM_E m}{r_f}$$

Plug in $r_i = 3R_E$, and $r_f = R_E + h$, where $h = 100$ km, we have

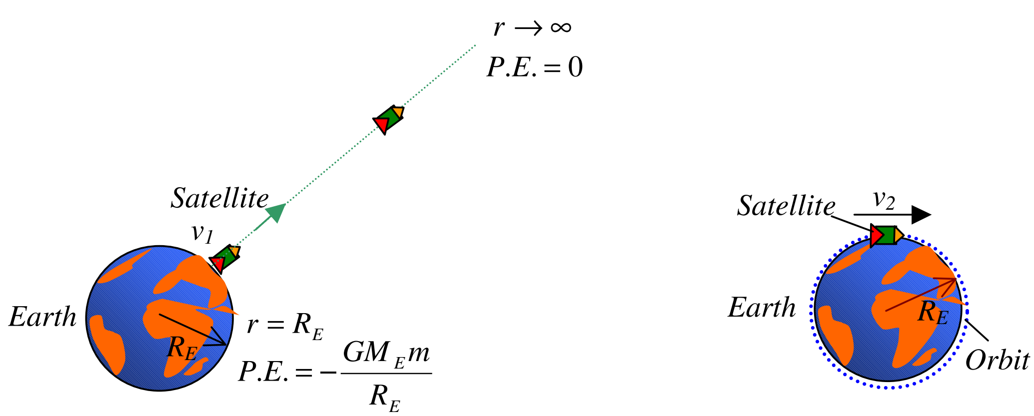
$$\frac{1}{2}mv_i^2 - \frac{GM_E m}{3R_E} = \frac{1}{2}mv_f^2 - \frac{GM_E m}{R_E + h}$$

That is $v_f^2 = v_i^2 - 2GM_E(\frac{1}{3R_E} - \frac{1}{R_E + h})$, where $v_i^2 = v^2 + \frac{GM_E}{3R_E}$.

Plug in the data for G, R_E, M_E and h, we obtain the velocity of the equipment when it enters the atmosphere, $v = 10.1\times10^3$m/s.

Example

If the minimum speed for a projectile to escape from the earth is v_1 and the orbital speed of a satellite circling close to the earth is v_2. What is the ratio of v_1 to v_2?



(a) (b)

Answer:

To obtain v_1, we have to consider the conservation of mechanical energy as shown in figure (a).

$$\frac{1}{2}mv_1^2 - \frac{GM_E m}{R_E} = 0,$$

where the term, $-\frac{GM_E m}{R_E}$, is the gravitational potential energy. Now, the escape velocity is given by $v_1^2 = \frac{2GM_E}{R_E}$.

On the other hands, the circular orbit (figure b) is maintained by the centripetal force which is in turn provided by the gravitational attraction

$$\frac{mv_2^2}{R_E} = \frac{GM_E m}{R_E^2}.$$

Hence, the orbital speed is given by $v_2^2 = \frac{GM_E}{R_E}$.

As a result, $\frac{v_1^2}{v_2^2} = 2$ and $v_1 : v_2 = \sqrt{2} : 1$.

9.7 Kepler's laws

Kepler announced his laws about planetary motion in 1609 for the first two laws and in 1619 for the third law.

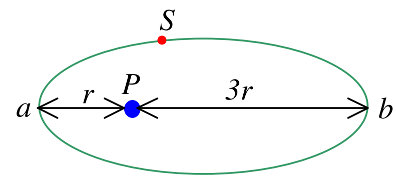
(a) Each planet moves in an ellipse which has the sun at one focus.

(b) The line joining the sun to the moving planet sweeps out equal areas in equal times.

(c) The squares of the times of revolution of the planets (i.e. their periodic times *T*) about the sun are proportional to the cubes of their mean distance *r* from it (i.e. $\frac{r^3}{T^2} = \text{constant}$).

Example

A satellite moves around a planet in an elliptical orbit. What is the ratio of the speed of the satellite at point a to that at point b?



Answer:

By the conservation of angular momentum, we have

$$I_a \omega_a = I_b \omega_b \, ,$$

where I_a and I_b are the moment of inertia of the system about P at positions a and b respectively.

Now, we can write $mr_a^2 (\frac{v_a}{r_a}) = mr_b^2 (\frac{v_b}{r_b})$,

which gives $\frac{v_a}{v_b} = \frac{r_b}{r_a} = \frac{3}{1}$. That is, $v_a : v_b = 3 : 1$.