Hong Kong Physics Olympiad Lesson 10

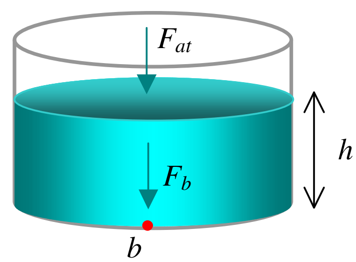
10.1 Static equilibrium in fluids: Pressure and depth

Consider a cylindrical container of cross-section area A, filled with fluid a height h. The top surface is at an atmospheric pressure $P_{at.}$ At the bottom of the container (point b), the downward force is F_{at} plus the weight of the fluid.

$$F_b = F_{at} + mg$$

$$P_b = \frac{F_b}{A} = \frac{F_{at}}{A} + \frac{mg}{A}$$

$$= P_{at} + \frac{\rho V g}{A}$$



where ρ is the density of the fluid.

Hence, we obtain $P_b = P_{at} + \rho g h$.

Remarks:

You may note that one day while swimming below the surface of a river, you let out a small bubble of air from your mouth. As the bubble rises toward the water surface, its diameter increases. The reason is simple, since the bubble rises the pressure in the surrounding water decreases. The volume of air bubble thus expands.

Example

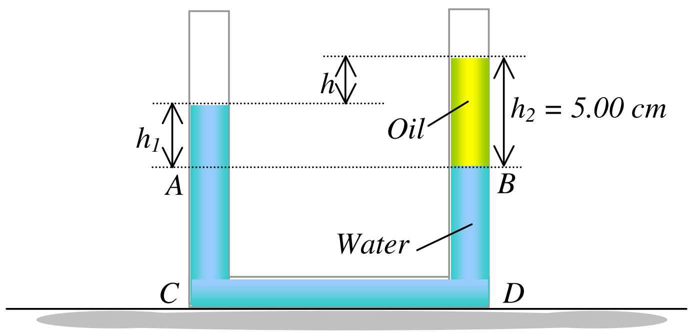
A U-shaped tube is filled mostly with water, but a small amount of vegetable oil has been added to one side. The density of the water is 1000 kg/m^3, and the density of the vegetable oil is 920 kg/m^3. If the depth of the oil is 5.00 cm, what is the difference in level h between the top of the oil on one side of the U and the top of the water on the other side?

Answer:

Note that the pressure at point A is the same as that at point B. Hence we can write

$$P_{atm} + \rho_1 g h_1 = P_{atm} + \rho_2 g h_2$$

That is $\rho_1 h_1 = \rho_2 h_2$, or $h_1 = \dfrac{\rho_2 h_2}{\rho_1}$.



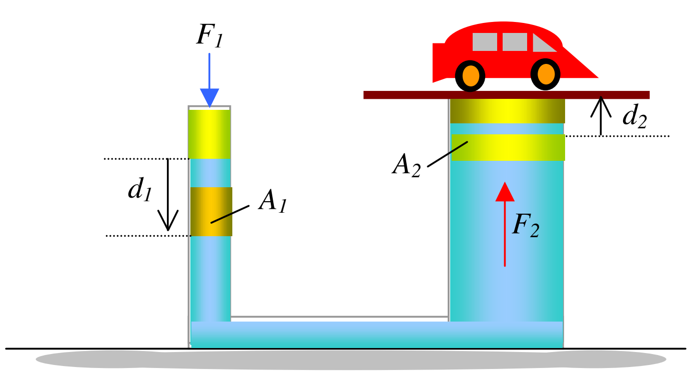
Plugging in the densities of water and oil as well as the height of oil in the U tube. We obtain $h_1 = 4.60$ cm.

Since $h = h_2 - h_1 = 5.00$ cm $- 4.60$ cm $=$ 0.40 cm.

10.2 Pascal's Principle

Recall that if the atmospheric pressure is P_{at}, the pressure at a depth h below the fluid surface is $P = P_{at} + \rho g h$. Suppose now, that the atmospheric pressure is replaced by a pressure $P_{at} + \Delta P$, the pressure at the depth h is $P = P_{at} + \Delta P + \rho g h$. Thus, by increasing the pressure at the top of the fluid by an amount, say, ΔP, we have increased it by the same amount everywhere in the fluid.

Pascal's Principle states that an external pressure applied to an enclosed fluid is transmitted unchanged to every point within the fluid. An example of Pascal's Principle is the hydraulic lift, which is sketched in the following figure. The cross-sectional area A_2 is larger than A_1. Suppose the piston 1 in the left cylinder is pushed



by a downward force F_1, the extra pressure other than the atmospheric pressure at the fluid surface is $\Delta P = \frac{F_1}{A_1}$. This extra pressure is transmitted everywhere in the fluid. At the right piston, that is piston 2, an extra pressure of the same amount is exerted on it. And the force acting upward on piston 2 is

$$F_2 = (\Delta P)A_2 .$$

Note that the greater the area of piston 2 the greater will be the upward force. As a result, it is a good machine for lifting objects. Substituting the increase in pressure in the above relation $F_2 = (\frac{F_1}{A_1})A_2 = (\frac{A_2}{A_1})F_1 > F_1$. For an incompressible fluid, the volume of fluid moved in the left cylinder should be the same as that in the right cylinder. One has $A_1 d_1 = A_2 d_2$. Hence, we obtain $F_2 = (\frac{A_2}{A_1})F_1 = (\frac{d_1}{d_2})F_1$ or we can conclude that the force is inversely proportional to the displacement of the piston.

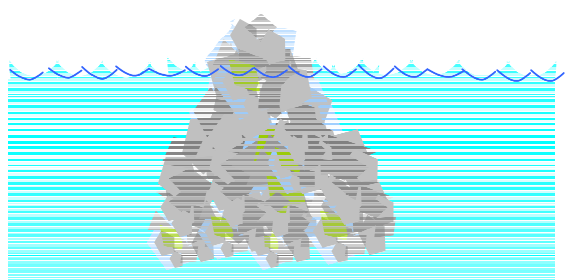
$$\frac{F_1}{F_2} = \frac{d_2}{d_1} .$$

10.3 Archimedes' Principle and buoyancy

A fluid exerts a net upward force on any object it surrounds. This is referred to as a buoyant force. Archimedes's Principle states that an object wholly or partially immersed in a fluid is buoyed up by a force equal in magnitude to the weight of the fluid displaced by the object.

Example

What fraction of the total volume of an iceberg is exposed?



Answer:

The weight of the iceberg is

$$W_i = \rho_i V_i g ,$$

where V_i and ρ_i are the volume and the density of the iceberg respectively.

The volume of the submerged portion of the iceberg relates to the buoyant force

$$F = \rho_w V_w g ,$$

where V_w and ρ_w are the volume and the density of water which is displaced by the iceberg.

At equilibrium, the upward and downward forces balance each other

$$F = W_i.$$

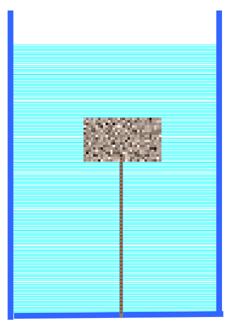
Hence, $\rho_w V_w g = \rho_i V_i g$ and we obtain the volume ratio

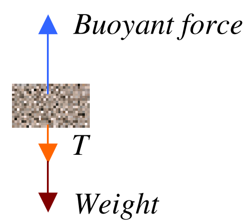
$$\frac{V_w}{V_i} = \frac{\rho_i}{\rho_w} .$$

Plugging in the densities of ice (917 kg/m^3) and water (1000 kg/m^3) into the above equation, we conclude that 8.3% volume of ice is above water.

Example

A piece of plastic with a density of 706 kg/m^3 is tied with a string to the bottom of a water-filled flask. The plastic is completely immersed, and has a volume of 8.00×10^{-6} m^3. What is the tension in the string?





Answer:

The upward force is the buoyant force = $\rho_w Vg$.

The downward forces are the tension of string plus the weight of plastic = $T + \rho_p Vg$.

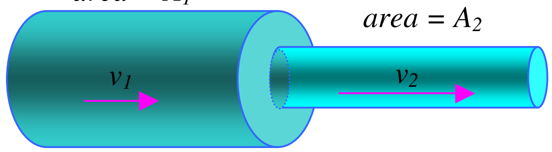
As the system is in equilibrium, the upward and downward forces balance each other e.g. $T + \rho_p Vg = \rho_w Vg$ which gives the tension of string

$$T = \rho_w Vg - \rho_p Vg .$$

Plugging in the densities of plastic and water, the volume of plastic and the gravitational acceleration 9.8 ms^{-2}. The tension of string is 0.0231 N.

10.4 Continuity of fluid flow

Imagine a fluid flows in a cylindrical pipe, the mass passing through the large pipe in a given time, Δt, must also flow past the small pipe in the same time.



The masses of fluid flow in the large pipe and the small pipe are

$$\Delta m_1 = \rho_1 \Delta V_1 = \rho_1 A_1 v_1 \Delta t \,,$$

and $$\Delta m_2 = \rho_2 \Delta V_2 = \rho_2 A_2 v_2 \Delta t$$

respectively.

As $\Delta m_1 = \Delta m_2$, we obtain $\rho_1 A_1 v_1 = \rho_2 A_2 v_2$. If the fluid is incompressible, $\rho_1 = \rho_2$, we thus obtain $A_1 v_1 = A_2 v_2$.

10.5 Bernoulli's equation

The pressure acting on a moving fluid does work on it that appears as a net change in the kinetic and / or potential energy of the system; that is,

$$\Delta W = \Delta KE + \Delta PE \,.$$

Consider the tube in the following figure, the pressure-force $F_1 = P_1 A_1$ acting on the fluid, pushing it in the direction of motion, does an amount of work on it, e.g. $F_1 \Delta l_1$. In the same time, the fluid external to the tube pushes to the left with a force $F_2 = P_2 A_2$, which is opposite in direction to the displacement. Here, the liquid in the tube is doing work pushing on the surrounding fluid, and so the work done on it is $F_2 \Delta l_2$. The net work done on the fluid sample is

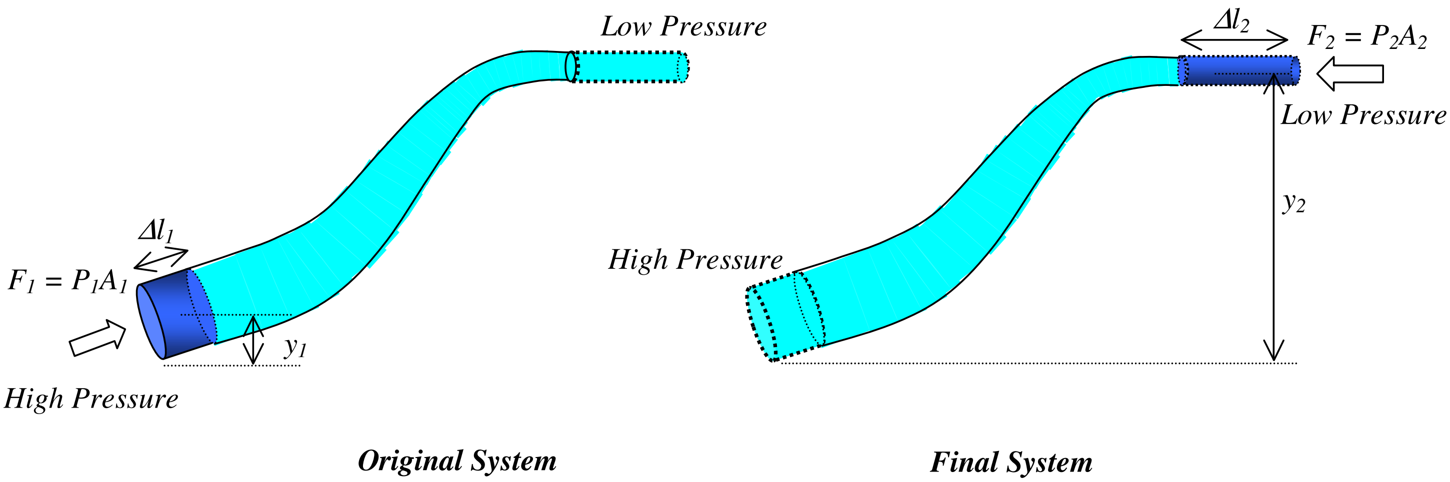
$$\begin{aligned}\Delta W &= F_1 \Delta l_1 - F_2 \Delta l_2 \\ &= P_1 A_1 \Delta l_1 - P_2 A_2 \Delta l_2 \\ &= P_1 A_1 (v_1 \Delta t) - P_2 A_2 (v_2 \Delta t) \\ &= \Delta V (P_1 - P_2)\end{aligned}$$

Since the amount of fluid moved in a time interval Δt is ΔV, which is given by

$$\Delta V = A_1 v_1 \Delta t = A_2 v_2 \Delta t \,.$$

Hence, we obtain the work done due to the pressure difference

$$\Delta W = \frac{\Delta m}{\rho}(P_1 - P_2)\,.$$



The change in kinetic energy of the fluid in the tube is

$$\Delta KE = \frac{1}{2}\Delta m(v_2^2 - v_1^2)\,.$$

The change in gravitational potential energy arises because the shifting of all the molecules in the tube,

$$\Delta PE = \Delta m g(y_2 - y_1)\,.$$

By using the conservation of energy, $\Delta W = \Delta KE + \Delta PE$, we can write

$$\frac{\Delta m}{\rho}(P_1 - P_2) = \frac{1}{2}\Delta m(v_2^2 - v_1^2) + \Delta m g(y_2 - y_1)\,.$$

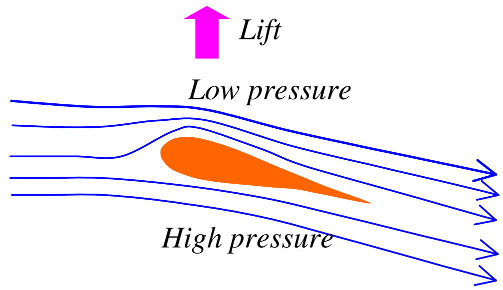
After simplification, we obtain the Bernoulli' equation

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho\, g\, y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho\, g\, y_2\,.$$

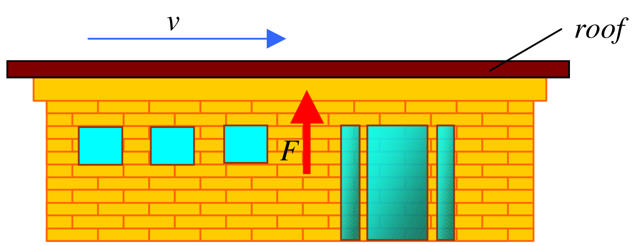
Each term has the dimension of energy per unit volume, or energy density. That is the the sum of pressure-energy density (P) arising from the internal forces on the moving fluid, the kinetic-energy density ($\frac{1}{2}\rho v^2$) and the potential-energy density ($\rho\, g\, y$) is a constant. The sum is sometimes called the net energy density of the fluid.

Remark:

If the pipe is horizontal, we have $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$. In other words, $P + \frac{1}{2}\rho v^2$ is a constant. Greater the flow rate of fluid at a certain point, lesser will be the pressure acting on it. As a practical example, the lifting force on an aerofoil is the result of the pressure difference exerted on it.

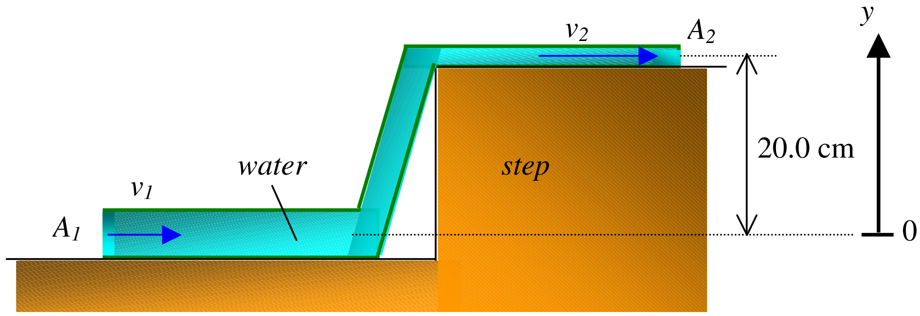


Another example is the net upward force on the roof when wind blows across the roof of a house. The lower pressure over the roof is accounted by the Bernoulli's equation.



Example

Water flows through a garden hose that goes up a step 20.0-cm height. If the water pressure is 143 kPa at the bottom of the step, what is its pressure at the top of the step? Given that the cross-sectional area of the hose on top of the step is half that at the bottom of the step and the speed of the water at the bottom of the step is 1.20 m/s.



Answer:

Assume that water is incompressible, the continuity equation states that

$$A_1 v_1 = A_2 v_2$$

Hence the velocity of water at the top of the step is $2 \times 1.20\, m/s = 2.40\, m/s$.

Using the Bernoulli's equation

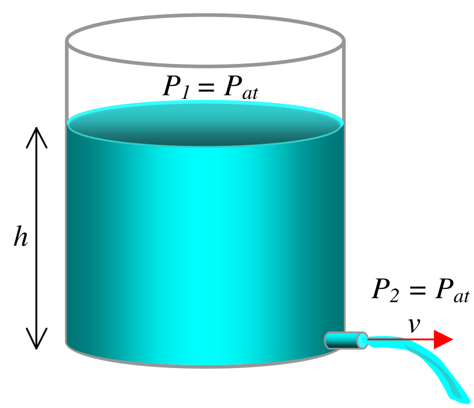
$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g\, y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g\, y_2\,,$$

which gives $P_2 = P_1 + \frac{1}{2}\rho(v_1^2 - v_2^2) + \rho g\,(y_1 - y_2)$.

Plugging in P_1 = 143 kPa, v_1 = 1.20 m/s, y_1 = 0 m, v_2 = 2.40 m/s, y_2 = 0.20 m, and ρ = 1000 kg/m^3, we obtain the pressure at the top of the step, P_2 = 130 kPa.

Example

Find the velocity of water as it emerges from the tip of a tank as shown in figure.



Answer:

This is a typical problem solved by the Bernoulli's equation. Plugging in data, e.g.

$P_1 = P_{at}$, $y_1 = h$ and $v_1 = 0$;

$P_2 = P_{at}$ and $y_2 = 0$,

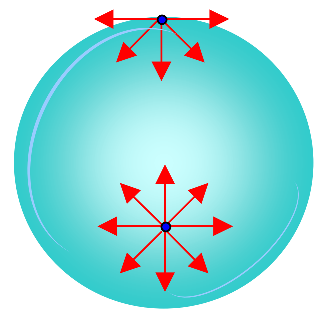
the equation $P_1 + \frac{1}{2}\rho v_1^2 + \rho g\, y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g\, y_2$

becomes $P_{at} + \frac{1}{2}\rho (0)^2 + \rho g\,(h) = P_{at} + \frac{1}{2}\rho v_2^2 + \rho g\,(0)$.

Hence $\rho g h = \frac{1}{2}\rho v_2^2$ and thus $v_2 = \sqrt{2gh}$.

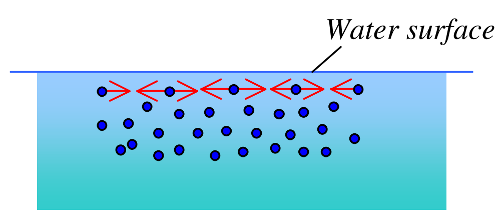
10.6 Surface tension

(a) *Liquid drop*



A molecule in the interior of a fluid experiences attractive forces of equal magnitude in all directions, giving a net force of zero. A molecule near the surface of the fluid experiences a net attractive force toward the interior of the fluid. This causes the surface to be pulled inward, resulting in a surface of minimum area. Consider a drop of liquid, surface tension plays an important role for the formation of it.

(b) *Water surface*



Molecules in the surface of a liquid are farther apart than those in the body of the liquid, i.e. the surface layer has a lower density than the liquid in bulk.

(c) *Molecular explanation*

The intermolecular forces in a liquid, like those in a solid, are both attractive and repelling and these balance when the spacing between molecules has its equilibrium value. However, when the separation is greater than the equilibrium value (r_0), the attractive force between molecules exceeds the repelling force. This is the situation with the more widely spaced surface layer molecules of a liquid. The attractive forces on either side due to their neighbours which puts them in a state of tension and thus the surface behaves like an elastic skin or membrane. When a small force is applied to the liquid surface it tends to stretch, resisting penetration.

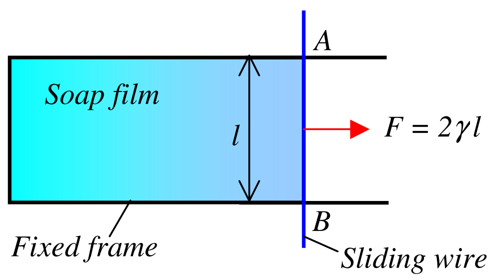
(d) *Definition of surface tension*

Consider a straight line of length l in the surface of a liquid. If the force acting at right angles to this line and in the surface is F, then the surface tension γ of the liquid is defined by

$$\gamma = \frac{F}{l}.$$

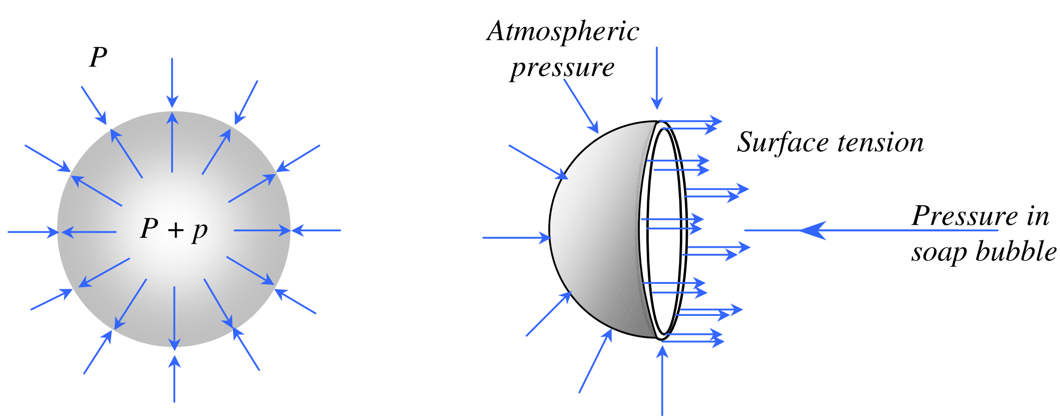
In words, γ is the force per unit length acting in the surface perpendicular to one side of a line in the surface. It is temperature dependent and has an unit of Nm^{-1}. At $20^{o}C$, for water $\gamma = 72.6 \times 10^{-3}$ Nm^{-1} and for mercury $\gamma = 465 \times 10^{-3}$ Nm^{-1}.

It should be noted that a thin film of soap or a bubble of soap has two surfaces, but the water drop has one surface. For example, the wire AB in the following figure is kept at rest by an external force F such that the surface tension force is balanced. As a film of soap has two surfaces and so the width of film contributing surface tension force is $2l$ and $F = 2\gamma l$.



10.7 Pressure difference across a curved surface

Consider a soap bubble in the following figure. The inward forces on the bubble are contributed by the atmospheric pressure P and the surface tension force. But these



forces are balanced by the force which is given by the pressure inside the bubble. As a result, this pressure exceeds the atmospheric pressure by p, and the magnitude of it is given by $P+p$. It is noted that the pressures inside ($P+p$) or outside (P) the bubble provide horizontal forces

$$P\pi r^2 + 2(2\gamma\pi r) = (P+p)\pi r^2$$

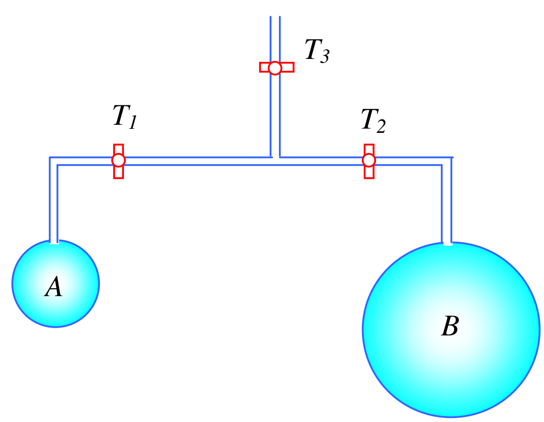
$$4\gamma\pi r = p\pi r^2$$

$$p = \frac{4\gamma}{r}.$$

Taking γ for a soap solution as 2.5×10^{-2} Nm^{-1}, the excess pressure inside a bubble of radius 1.0 cm is $p = \frac{4\times 2.5\times 10^{-2}}{1.0\times 10^{-2}}\frac{Nm^{-1}}{m} = 10\,Pa$.

Example

If the soap bubbles of different radii are blown separately using the apparatus as shown in figure. Now, taps T_1 and T_2 are opened, what is your observation?

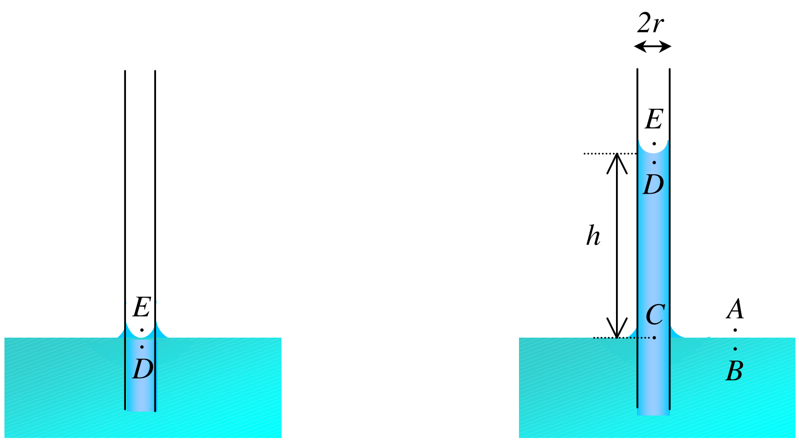


Answer:

The smaller bubble A will decrease its size and the larger bubble expands. Equilibrium will be attained when the two bubbles have the same radii. That bubble A becomes a curved film and bubble B has a larger radius.

10.8 Capillary rise formula

Consider a fluid in a capillary tube, the fluid rises up to a height h due to the pressure difference across the curved fluid surface in the tube. Assume that the meniscus (fluid surface) is in the form of spherical shape and the pressure difference at E and D is



$p = \dfrac{2\gamma}{r}$. Of course, p_E is greater than p_D, since E is curved downward. The point C is at a depth h under D, the pressure at C, i.e. p_C, exceeds p_D by ρgh.

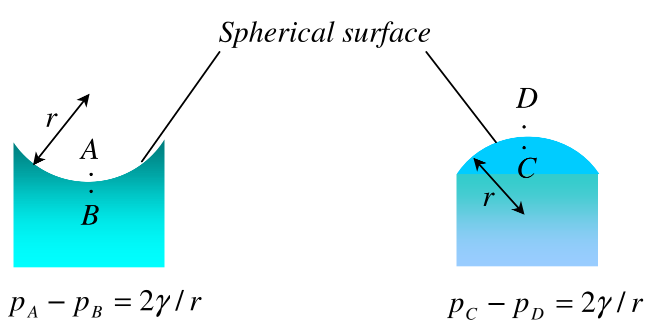
$$\begin{aligned} p_C &= p_D + \rho gh \\ &= p_E - \frac{2\gamma}{r} + \rho gh \end{aligned}$$

As p_E is at an atmospheric pressure, $p_C = p_{atm} - \dfrac{2\gamma}{r} + \rho gh$. At equilibrium, $p_C = p_A = p_B = p_{atm}$. Hence we can write

$$p_C = p_{atm} - \frac{2\gamma}{r} + \rho gh = p_{atm} \qquad \text{or} \qquad \rho gh = \frac{2\gamma}{r},$$

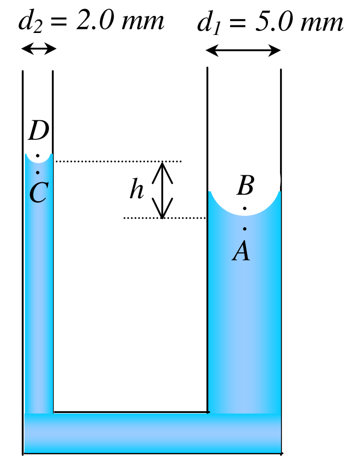
The height of fluid h is given by $h = \dfrac{2\gamma}{r\rho g}$.

Remarks:



Example

A U-tube with different limb diameters is shown in figure. At equilibrium, water with level difference is observed in the two limbs. Given that the surface tension of water is 7.0×10^{-2} Nm^{-1} and its density is 1000 kgm^{-3}. If the contact angle is zero between water surface and the limbs, what is the difference in water level.



Answer:

Since the surfaces of water is curved downward, we conclude that

$p_B > p_A$ and $p_D > p_C$.

Mathematically, we have

$$p_B - p_A = \frac{2\gamma}{r} = \frac{2(7.0\times10^{-2})}{5.0\times10^{-3}/2} = 56\,Pa\,,$$

and $$p_D - p_C = \frac{2\gamma}{r} = \frac{2(7.0\times10^{-2})}{2.0\times10^{-3}/2} = 140\,Pa\,.$$

But, $p_B = p_D = p_{atm}$, the pressures p_A and p_C can be rewritten as

$$p_A = (p_{atm} - 56)\,Pa\,,$$

and $$p_C = (p_{atm} - 140)\,Pa\,.$$

Now, the difference between p_A and p_C is due to the water column of height h,

$$p_A - p_C = \rho g h\,.$$

Plugging in the expressions of p_A and p_C, we obtain $(p_{atm} - 56) - (p_{atm} - 140) = \rho g h$,

which gives $\rho g h = 84\,Pa$.

The difference in water levels is given by

$$h = \frac{84\,Pa}{\rho g} = \frac{84\,Pa}{(1000\,kgm^{-3})(9.8\,ms^{-2})} = 8.6\,mm\,.$$