Answers
Part I

Q1. The plane should follow the parabola 飛機須沿拋物線運動。

$x = v_0 t\cos\theta,\ y = v_0 t\sin\theta - \frac{1}{2}gt^2$ **(3 points)**

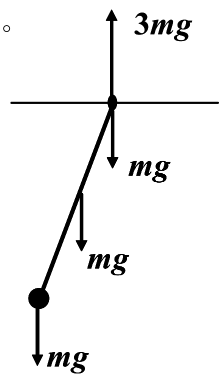
Q2 (6 points)

$I\ddot{\omega} = T$

The center of the rod will not move in the horizontal direction 杆中心在水平方向不動。

$I = \frac{ml^2}{12} + 2m(\frac{l}{2})^2 = \frac{7}{12}ml^2$ (2 points)

There are two ways to find the torque. 找力距的方法有兩種。



Method-1 方法-1

The forces acting upon the rod are shown. The torque to the center of the rod is 由如圖力的分析，可得

$T = -(2mg + mg)\frac{l}{2}\theta = -mg\frac{3l}{2}\theta$. (2 points)

Method-2 方法-2

Given a small angle deviation θ from equilibrium, the potential energy is 給定一個角度的小位移θ ，勢能爲

$U = mg\frac{3l}{2}(1-\cos\theta) \approx mg\frac{3l}{4}\theta^2$.

$T = -\frac{\partial U}{\partial\theta} = -mg\frac{3l}{2}\theta$ (2 points)

Finally, 最後得 $\frac{7ml^2}{12}\ddot{\theta} = \frac{3mgl}{2}\theta \Rightarrow \omega = \sqrt{\frac{18g}{7l}}$.(2 points)

Q3 (6 points)

(a) The bound current density on the disk edge is 盤邊的束縛電流密度爲

$K = -\vec{M}\times\vec{n} = -M$, (1 point)

The bound current is 束縛電流 $\Rightarrow I = Jd = -Md$, (1 point)

The B-field is 磁場爲 $B(z=h) = \frac{\mu_0 R^2 I}{2(R^2+h^2)^{\frac{3}{2}}} = -\frac{\mu_0 R^2 Md}{2(R^2+h^2)^{\frac{3}{2}}}$ (1 point)

(b) The bound current density is $K = -\vec{M} \times \vec{n} = -M$, which is on the side wall of the cylinder. (1 point)

柱側面上的束縛電流密度爲 $K = -\vec{M} \times \vec{n} = -M$

The problem is then the same as a long solenoid. Take a small Ampere loop we get $B = \mu_0 K = \mu_0 M$ inside;

爲求一長線圈的磁場，取一小閉合路徑，得介質內 $B = \mu_0 K = \mu_0 M$ (1 point)

Outside 介質外 $B = 0$ (1 point)

Q4 (5 points)

Each unit charge in the slab experiences the Lorentz force $-vB\,\vec{y}_0$. (1 point)

The problem is then the same as a dielectric slab placed between two parallel conductor plates that carry surface charge density $\pm\sigma$, and $\frac{\sigma}{\varepsilon_0} = vB$. In such case, the electric displacement is $D = \sigma$. $P = D - \varepsilon_0 E = D - \frac{D}{\varepsilon} = \varepsilon_0 vB(\frac{\varepsilon - 1}{\varepsilon})$. (2 point)

介質內單位電荷受力 $-vB\,\vec{y}_0$ 。問題變成兩電荷面密度爲 $\pm\sigma$, and $\frac{\sigma}{\varepsilon_0} = vB$ 的導電板間充滿介質。因此 $D = \sigma$. $P = D - \varepsilon_0 E = D - \frac{D}{\varepsilon} = \varepsilon_0 vB(\frac{\varepsilon - 1}{\varepsilon})$.

Finally, the bound surface charge is $\sigma_b = P = \varepsilon_0 vB(\frac{\varepsilon - 1}{\varepsilon})$. The upper surface carries positive bound charge, and the lower surface carries negative charge. (1 point)

最後得束縛電荷密度 $\sigma_b = P = \varepsilon_0 vB(\frac{\varepsilon - 1}{\varepsilon})$ ，上表面帶正電，下表面帶負電。

The electric field is $\vec{E} = \frac{\sigma_b}{\varepsilon_0}\vec{y}_0 = vB(\frac{\varepsilon - 1}{\varepsilon})\vec{y}_0$, which is along the y-direction (opposite to the Lorentz force). (1 point)

電場為$\vec{E}=\dfrac{\sigma_b}{\varepsilon_0}\vec{y}_0=vB(\dfrac{\varepsilon-1}{\varepsilon})\vec{y}_0$，與 Lorentz 力方向相反。

Q5. (10 points)

(a) Because of the spherical symmetry, the E-field and the current density $\vec{J}$ are all along the radial direction. In steady condition, the electric current I through any spherical interfaces must be equal. Since the area of the sphere is proportional to r^2, $\vec{J}$ must be proportional to $1/r^2$. So let $\vec{J}=\dfrac{K}{r^2}\hat{r}$, where K is a constant to be determined, and the expression holds in both media. (1 point)

由對稱性可知，電場和電流密度$\vec{J}$須沿半徑方向。穩態時，流過每個包住球心的球面的電流相等，因此$\vec{J}$與 $1/r^2$ 成正比。設在兩介質裏$\vec{J}=\dfrac{K}{r^2}\hat{r}$，$K$為待定常數。

In medium-1 介質-1, $E_1=\dfrac{1}{\sigma_1}J=\dfrac{1}{\sigma_1}\dfrac{K}{r^2}$, and the voltage drop from R_1 to R_2 is

$$V_1=\frac{1}{\sigma_1}K(\frac{1}{R_1}-\frac{1}{R_2}) \qquad \text{(1 point)}$$

介質-1, $E_1=\dfrac{1}{\sigma_1}J=\dfrac{1}{\sigma_1}\dfrac{K}{r^2}$，從 R_1 到 R_2 的電壓為$V_1=\dfrac{1}{\sigma_1}K(\dfrac{1}{R_1}-\dfrac{1}{R_2})$

Likewise, in medium-2, $E_2=\dfrac{1}{\sigma_2}\dfrac{K}{r^2}$, and the voltage drop from R_2 to R_3 is

$$V_2=\frac{1}{\sigma_2}K(\frac{1}{R_2}-\frac{1}{R_3}). \qquad \text{(1 point)}$$

同樣，在介質-2, $E_2=\dfrac{1}{\sigma_2}\dfrac{K}{r^2}$，從 R_2 到 R_3 的電壓為$V_2=\dfrac{1}{\sigma_2}K(\dfrac{1}{R_2}-\dfrac{1}{R_3})$

The total voltage drop between R_1 and R_3 is 總電壓 $V=V_1+V_2$.

So 得 $\frac{1}{K}=\frac{1}{V}\left[\frac{1}{\sigma_1}(\frac{1}{R_1}-\frac{1}{R_2})+\frac{1}{\sigma_2}(\frac{1}{R_2}-\frac{1}{R_3})\right]$

The current is 電流為 $I=4\pi R_1^2 J(R_1)=4\pi K$. (1 point)

The electric displacement in media are $D_{1,2}=\frac{\varepsilon_{1,2}}{\sigma_{1,2}}J$, so the charge on the inner, outer, and boundary shells are $4\pi K\frac{\varepsilon_1}{\sigma_1}$, $-4\pi K\frac{\varepsilon_2}{\sigma_2}$, and $4\pi K\left(\frac{\varepsilon_2}{\sigma_2}-\frac{\varepsilon_1}{\sigma_1}\right)$, respectively.

介質-1 裏電位移 $D_{1,2}=\frac{\varepsilon_{1,2}}{\sigma_{1,2}}J$，因此在各面上的電荷為 $4\pi K\frac{\varepsilon_1}{\sigma_1}$, $-4\pi K\frac{\varepsilon_2}{\sigma_2}$, $4\pi K\left(\frac{\varepsilon_2}{\sigma_2}-\frac{\varepsilon_1}{\sigma_1}\right)$ 。

(b) Due to symmetry, the electric field is of the form $\vec{E}=\frac{K}{r^2}\hat{r}$,

so $V=K\left(\frac{1}{R_1}-\frac{1}{R_3}\right)$, and $K=V\cdot\frac{R_3-R_1}{R_1R_3}$. (2 point)

由對稱性可知，電場為 $\vec{E}=\frac{K}{r^2}\hat{r}$ ，因此電壓為 $V=K\left(\frac{1}{R_1}-\frac{1}{R_3}\right)$ ，得 $K=V\cdot\frac{R_3-R_1}{R_1R_3}$ 。

The current densities in the two hemispheres are $\vec{J}_1=\sigma_1\vec{E}$ and $\vec{J}_2=\sigma_2\vec{E}$. The total current is $I=2\pi R_1^2\cdot J_1(R_1)+2\pi R_1^2\cdot J_2(R_1)=2\pi K(\sigma_1+\sigma_2)$. (2 point)

上下半部的電流密度為 $\vec{J}_1=\sigma_1\vec{E}$ ，$\vec{J}_2=\sigma_2\vec{E}$ 。 總電流為

$$I=2\pi R_1^2\cdot J_1(R_1)+2\pi R_1^2\cdot J_2(R_1)=2\pi K(\sigma_1+\sigma_2).$$

The total free charge is $Q_1=2\pi\varepsilon_0\varepsilon_1R_1^2\cdot E_1(R_1)=2\pi K\varepsilon_0\varepsilon_1$ on the upper half and $Q_2=2\pi K\varepsilon_0\varepsilon_2$ on the lower half of the inner shell. On the outer shell the charges are negative of the corresponding ones of the inner shell. (1 point)

內球面上半部總自由電荷為 $Q_1 = 2\pi\varepsilon_0\varepsilon_1 R_1^2 \cdot E_1(R_1) = 2\pi K\varepsilon_0\varepsilon_1$ ，下半部總自由電荷為 $Q_2 = 2\pi K\varepsilon_0\varepsilon_2$ 。外球面的電呵與內球面相反。

Q6. (12 points)

a) $P_h - P_{h+dh} = \rho g dh \Rightarrow \dfrac{dP}{dh} = -\rho g$ (1 point)

$PV^{\frac{7}{5}} = \text{Constant} \Rightarrow P = C\rho^{\frac{7}{5}} \Rightarrow \rho = (\dfrac{P}{C})^{\frac{5}{7}}, where(C = \dfrac{P_0}{\rho_0^{\frac{7}{5}}})$ (1 point)

Combine these two equations, 合併兩式得

$\dfrac{dP}{dh} = -(\dfrac{P}{C})^{\frac{5}{7}} g \Rightarrow P = P_0(1 - \dfrac{2\rho_0 gh}{7P_0})^{\frac{7}{2}}$ (2 point)

$\rho = \rho_0(1 - \dfrac{2\rho_0 gh}{7P_0})^{\frac{5}{2}}$, and $T = T_0(1 - \dfrac{2\rho_0 gh}{7P_0})$. (2 point)

b) $TV^{\frac{2}{5}} = \text{Constant}, \dfrac{T}{\rho^{\frac{2}{5}}} = \text{Constant}$ (1 point)

The density at 40°C is $\dfrac{\rho^{\frac{2}{5}}}{313} = \dfrac{1.18^{\frac{2}{5}}}{293} \Rightarrow \rho = 1.18 \times (\dfrac{313}{293})^{\frac{5}{2}}$ (1 point)

40°C 的空氣密度為 $\dfrac{\rho^{\frac{2}{5}}}{313} = \dfrac{1.18^{\frac{2}{5}}}{293} \Rightarrow \rho = 1.18 \times (\dfrac{313}{293})^{\frac{5}{2}}$

The fraction of water vapor at 40°C at sea level 40°C 的水蒸汽分壓為,

$\eta_1 = \dfrac{55.35}{760} \times 90\%$ (1 point)

The fraction of water at 5°C at high altitude 5°C 的水蒸汽分壓為,

$\eta_2 = \dfrac{6.5}{760(\dfrac{278}{313})^{\frac{7}{2}}}$ (1 point)

Rain 下雨量

$$= (\eta_1 - \eta_2) \times \rho \times V = (\frac{55.35}{760} \times 90\% - \frac{6.5}{760(\frac{278}{313})^{\frac{7}{2}}}) \times 1.18 \times (\frac{313}{293})^{\frac{5}{2}} = 0.07kg \quad \text{(1 point)}$$

高度：$278 = 313(1 - \frac{2 \times 1.18 \times 9.8 \times h}{7 \times 1.03 \times 10^5}) \Rightarrow h = 3500m$ (1 point)

Q7. (8 points)

i) $\vec{p} \times \vec{E}$ (1 point)

ii) $\vec{P} \times \vec{E}$ (1 point)

iii) $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \Rightarrow \vec{P} = (\varepsilon - 1)\varepsilon_0 \vec{E} \Rightarrow \vec{P} \times \vec{E} = 0$ (1 point)

iv) $\vec{P} = \vec{D} - \varepsilon_0 \vec{E} = \varepsilon_0[(\varepsilon_X - 1)E_X \vec{X}_0 + (\varepsilon_Y - 1)E_Y \vec{Y}_0]$ (1 point)

$\overline{T} = \frac{1}{2}\text{Re}(\vec{P} \times \vec{E}^*) = \frac{1}{2}\text{Re}(\vec{D} \times \vec{E}^*) = \frac{1}{2}\varepsilon_0 E_0^2 (\varepsilon_X - \varepsilon_Y)\cos(\Delta kz)$, (1 point)

Where 其中 $\Delta k \equiv (\sqrt{\varepsilon_X} - \sqrt{\varepsilon_Y})\frac{\omega}{c}$. (1 point)

v) $\overline{T} = \frac{1}{2}\varepsilon_0(\varepsilon_X - \varepsilon_Y)E_0^2 \int_0^d \cos(\Delta kz)dz = \frac{1}{2\Delta k}\varepsilon_0(\varepsilon_X - \varepsilon_Y)E_0^2 \sin(\Delta kd)$. (1 point)

Maximum occurs when 最大值在 $\Delta kd = \frac{\pi}{2} \Rightarrow d = \frac{\pi}{2\Delta k}$. (1 point)

Part II

Q1. (6 points)

a) $k_x a = n\pi \Rightarrow k_x = \frac{n\pi}{a}, n = 1,2,3,.......$ (1 point)

b) $k_y 2b\pi = 2m\pi \Rightarrow k_x = \frac{m}{b}, m = 1,2,3,......$ (3 points)

c) $W = \frac{1239}{2\pi}\sqrt{k_x^{\ 2} + k_y^{\ 2}} \Leftrightarrow \frac{1239}{2\pi}\sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2}{b^2}} = 10^{12}$

$\Rightarrow \frac{1239}{2\pi}\frac{m}{b} < 10^{12} \Rightarrow b > \frac{1239}{\pi} \times 10^{-12} \approx 2 \times 10^{-10}\, nm$ (2 points)

Q2. (22 points)

i) $I = m_2 l^2 + \frac{m_1}{3} l^2$ (2 points)

$I\dot{\omega} = F(t)\cdot l \Leftrightarrow (m_2 l^2 + \frac{m_1}{3} l^2)\ddot{\theta} = -\theta k l^2 \Rightarrow \omega_0 = \sqrt{\frac{k}{m_2 + \frac{m_1}{3}}}$ (2 points)

ii) $(m_2 l + \frac{m_1 l}{3})\ddot{x} = F_1 \cos(\omega_1 t) l - xkl$ (2 points)

$x = A_1 \cos(\omega_1 t + \phi_1)$

$\Rightarrow -(\frac{m_1}{3} + m_2) A_1 \omega_1^{\ 2} \cos(\omega_1 t + \phi_1) = F_1 \cos(\omega_1 t) - A_1 \cos(\omega_1 t + \phi_1) k$

$\Rightarrow \phi_1 = 0, A_1 = \frac{F_1}{k - (\frac{m_1}{3} + m_2)\omega_1^{\ 2}}$ (2 points)

iii) $(m_2 + \frac{m_1}{3})\ddot{x} = F_1 \cos(\omega_1 t) + F_2 \cos(\omega_2 t) - xk$ (2 points)

$x = A_1 \cos\omega_1 t + A_2 \cos\omega_2 t$

$\Rightarrow A_1 = \frac{F_1}{k - (\frac{m_1}{3} + m_2)\omega_1^{\ 2}}, A_2 = \frac{F_2}{k - (\frac{m_1}{3} + m_2)\omega_2^{\ 2}}$ (2 points)

iv) $x = \sum^{n} A_n \cos\omega_n t \quad \Rightarrow A_n = \frac{F_n}{k - (\frac{m_1}{3} + m_2)\omega_n^{\ 2}}$ (2 points)

v) $V_{out} = V_0 e^{i\omega t} = \frac{R + iL\omega}{\frac{1}{i\omega C} + R + iL\omega} V_0 e^{i\omega t} = \frac{iR\omega C - L\omega^2 C}{1 - L\omega^2 C + iR\omega C} V_0 e^{i\omega t}$

$$|V_{out}|^2 = V_0^2 \frac{(R\omega C)^2 + (L\omega^2 C)^2}{(1 - L\omega^2 C)^2 + (R\omega C)^2} \qquad \text{(1 point)}$$

To make the denominator minimum, we should have 使分母最小

$1 - LC\omega^2 = 0 \Rightarrow L = \dfrac{1}{C\omega^2}$ (1 point)

(vi) For a given L, only the signal with $\omega_n = \sqrt{\dfrac{1}{CL}}$ in the answer of (iv) can pass through the filter, (1 point) and the output is proportional to A_n. (1 point) By varying L one selects different ω_n, and the output is proportional to the selected A_n. (1 point) From (iv), the maximum A_n is the one when $k = (\dfrac{m_1}{3} + m_2)\omega_n^{\ 2}$. (1 point) So as L is varied, one finds a particular L_{max} at which the signal peaks, and $k = (\dfrac{m_1}{3} + m_2)CL_{max}$. (1 point)

給定 L, 則只有頻率爲 $\omega_n = \sqrt{\dfrac{1}{CL}}$ 的信號可通過濾波器, (1 point) 其大小正比於 A_n. (1 point) L 的變化等於選擇不同的 ω_n 的信號. (1 point) 由(iv)知, $k = (\dfrac{m_1}{3} + m_2)\omega_n^{\ 2}$ 時 A_n 最大. (1 point) 因此可得使輸出信號最大的電感 L_{max}，並有 $k = (\dfrac{m_1}{3} + m_2)CL_{max}$.

Correct sketch is a flat line with a peak at L_{max} 正確的簡圖是一平線，在 L_{max} 有一尖峰 (1 point)

Q3. (22 points)

i) Only the charge $q = \dfrac{r^3}{R^3}Ze$ that is inside the sphere r will have a non-zero net force on the nucleus. (2 points)

只有 $q = \dfrac{r^3}{R^3}Ze$ 這麼多的負電荷對原子核的合力不爲零。

$$\frac{1}{4\pi\varepsilon_0}\frac{Zeq}{r^2} = ZeE_0 \Rightarrow \frac{1}{4\pi\varepsilon_0}\frac{rZe}{R^3} = E_0 \Rightarrow r = \frac{4\pi\varepsilon_0 E_0 R^3}{Ze} \qquad \text{(2 points)}$$

ii) $Zm_e\ddot{r} = ZeE(t) - \frac{Ze}{4\pi\varepsilon_0}\times\frac{rZe}{R^3}$ (2 points)

$E(t) = A\cos(\omega t), r = B\cos(\omega t+\varphi)$ (1 point)

$$\Rightarrow -Zm_eB\omega^2\cos(\omega t+\varphi) = ZeA\cos(\omega t) - \frac{Z^2e^2}{4\pi\varepsilon_0}\times\frac{B\cos(\omega t+\varphi)}{R^3}$$

$$\Rightarrow \varphi = 0, B = \frac{eA}{\frac{Ze^2}{4\pi\varepsilon_0}\cdot\frac{1}{R^3} - m_e\omega^2}$$ (2 points)

$$\Rightarrow p = Zer = \frac{Ze^2A}{\frac{Ze^2}{4\pi\varepsilon_0}\cdot\frac{1}{R^3} - m_e\omega^2}\cos(\omega t)$$ (1 point)

iii) $P(t) = Np = \frac{NZe^2}{\frac{Ze^2}{4\pi\varepsilon_0}\cdot\frac{1}{R^3} - m_e\omega^2}E(t)$ (2 points)

iv) $P = CE_{ext} = (\varepsilon-1)\varepsilon_0E_{total} = (\varepsilon-1)\varepsilon_0(E_{ext}+E_{self})$ (2 points)

The electric field E in a uniform polarized ball with polarization vector P can be calculated by considering the surface bound charge $P\cos\theta$ distributed on the ball.

均勻極化球內的電場 E 可由球面的束縛電荷 $P\cos\theta$ 求得。

$$E_{self} = \frac{1}{4\pi\varepsilon_0}\iint\frac{P\cos\theta(-\cos\theta)}{R^2}R^2\sin\theta d\theta d\varphi = -\frac{P}{3\varepsilon_0}$$ (2 points)

$E_{self} = -\frac{P}{3\varepsilon_0} = \frac{1}{3}CE_{ext}$ (1 point)

$$\Rightarrow CE_{ext} = (\varepsilon-1)\varepsilon_0(E_{ext} - \frac{CE_{ext}}{3\varepsilon_0})$$

$$\Rightarrow 3\varepsilon_0\frac{\varepsilon-1}{\varepsilon+2} = C = \frac{NZe^2}{\frac{Ze^2}{4\pi\varepsilon_0}\cdot\frac{1}{R^3} - m_e\omega^2}$$

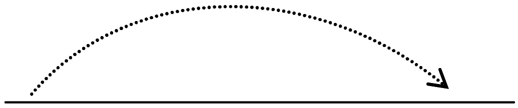
$$\Rightarrow K(\omega) = C = \frac{NZe^2}{\frac{Ze^2}{4\pi\varepsilon_0}\cdot\frac{1}{R^3} - m_e\omega^2} = \frac{N(Ze)^2}{Zm_e(\omega_0^2-\omega^2)},$$

where $\omega_0^2 \equiv \dfrac{4\pi\varepsilon_0 R^3}{Zm_e(Ze)^2}$ is the resonant frequenc of an atom (2 points)

and $n = 2$

$\omega_0^2 \equiv \dfrac{4\pi\varepsilon_0 R^3}{Zm_e(Ze)^2}$ 是原子的共振頻率。

v) For air K is very small and $\varepsilon(\omega)$ is close to 1.



So the L-L relation becomes

$\varepsilon(\omega) - 1 = \dfrac{1}{\varepsilon_0} K(\omega)$ (1 point)

Air density decreases with height. So refractive index decreases with height. (1 point)

Light rays are bent in such condition. (1 point)

空氣的$\varepsilon(\omega)$接近 1，K很小，上述結果簡化爲$\varepsilon(\omega) - 1 = \dfrac{1}{\varepsilon_0} K(\omega)$。空氣的密度隨高度減小，因此折射率也減小，形成如圖所示的光線彎曲。