第六届泛珠物理竞赛简单解答

Part-1 卷-1

Q1. 题-1

(i) Conservation of energy 能量守恒

$$E_n = E_0 - E_1 - \frac{1}{2}M_B v_{B1}^2$$

Conservation of momentum 动量守恒

$$M_A v_{A0}\hat{x} = M_A v_{A1}\hat{y} + M_B \vec{v}_{B1}$$

$$\vec{v}_{B1} = \frac{M_A}{M_B}\left(\sqrt{\frac{2E_0}{M_A}}\hat{x} - \sqrt{\frac{2E_1}{M_A}}\hat{y}\right)$$ (1 point)

So we have 得

$$E_n = E_0 - E_1 - \frac{M_A}{M_B}(E_0 + E_1)$$ (1 point)

(ii) For the capacitance: Gauss law 求电容，利用高斯定理

$$\int \nabla \cdot \vec{E}\, d^3x = \frac{Q}{\epsilon}$$

$$\vec{E} = \frac{Q}{4\pi\epsilon r^2}\hat{r}$$

$$V = -\int_a^b \vec{E}\cdot\hat{r}dr = \frac{Q}{4\pi\epsilon}\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$C = \frac{Q}{V} = 4\pi\epsilon\frac{ab}{b-a}$$ (1 point)

On the resistance: Ohm’s law 求电阻，利用 Ohm 定理

$$R = \frac{1}{\sigma}\int_a^b \frac{dr}{4\pi r^2} = \frac{1}{4\pi\sigma}\frac{b-a}{ab}$$ (1 point)

(iii) Linearly polarized along $\hat{x}_0$ 沿$\hat{x}_0$方向的线偏振 (1 point)

Left-handed circularly polarized: $\vec{E} = (\hat{x}_0 + i\hat{y}_0)E_0 e^{ikz-i\omega t}$ 左旋圆偏振 (1 point)

(iv)

$\vec{B} = \sqrt{\mu\epsilon}\hat{z}_0 \times \vec{E} = \sqrt{\mu\epsilon}E_0\hat{y}_0 e^{ikz-i\omega t}$ (1 point)

$\vec{S} = \frac{1}{2}\vec{E} \times \vec{H}^* = \frac{1}{2}\hat{z}_0\sqrt{\frac{\epsilon}{\mu}}E_0^2$ (1 point)

Replace μ by μ_0, ε by ε_0 for vacuum. 在真空中 μ 用 μ_0 代, ε 用 ε_0 代。

(v):

$\phi = \omega t - \vec{k}\cdot\vec{x} = \omega' t' - \vec{k}'\cdot\vec{x}'$, $\omega = ck_0$ $\omega' = ck_0'$

$$\begin{cases} k_0' = \gamma\left(k_0 - \vec{\beta}\cdot\vec{k}\right) \\ k_P' = \gamma\left(k_P - \beta k_0\right) \quad \vec{\beta} = \frac{\vec{v}}{c},\ \gamma = \left(1-\beta^2\right)^{-1/2} \\ k_\perp' = k_\perp \end{cases}$$ (1 point)

$\left|\vec{k}\right| = k_0$, $\omega = ck_0$ $\left|\vec{k}'\right| = k_0'$, $\omega' = ck_0'$

$\omega' = \gamma\omega\left(1-\beta\cos\theta\right)$

For $\theta = 0$, $\omega' = \gamma\omega(1-\beta)$ (1 point)

And for $\theta = \pi/2$, $\omega' = \gamma\omega$. (1 point)

Q2

(a) Using Newton's second law, 利用牛顿第二定律

$\frac{GMm}{r^2} = m\frac{v^2}{r}$, ➔ $v = \sqrt{\frac{GM}{r}}$

Kinetic energy 动能: $K = \frac{1}{2}mv^2 = \frac{GMm}{2r}$ (1 point)

(b) Let r_f be the further distance of the elliptical orbit from Earth. Let v_f be the velocity of this orbit at this distance. After the firing of the rocket, the velocity of the spacecraft becomes $\sqrt{1.3}v$.

令 r_f 为椭圆轨道离地球的最远点，v_f 为在该点飞船的速率。

Using the conservation of angular momentum,角动量守恒

$mr\sqrt{1.3}v = mr_f v_f$ (1) (1 point)

Initial total energy 初始总能量:

$K + U = 1.3\frac{GMm}{2r} - \frac{GMm}{r} = -0.35\frac{GMm}{r}$

Final total energy 现在总能量:

$$K+U=\frac{1}{2}mv_f^2-\frac{GMm}{r_f}$$

Using the conservation of energy, 能量守恒

$$\frac{1}{2}mv_f^2-\frac{GMm}{r_f}=-0.35\frac{GMm}{r} \qquad (2) \qquad \text{(1 point)}$$

Using (1) to eliminate v_f, 用(1)将 v_f代掉,

$$\frac{1}{2}mv^2\frac{1.3r^2}{r_f^2}-\frac{GMm}{r_f}=-0.35\frac{GMm}{r}$$

From the result of (a) 由(a)得,

$$\frac{GMm}{2r}\frac{1.3r}{r_f^2}-\frac{GMm}{r_f}=-0.35\frac{GMm}{r}$$

$$0.65\left(\frac{r}{r_f}\right)^2-\frac{r}{r_f}+0.35=0 \qquad \text{(1 point)}$$

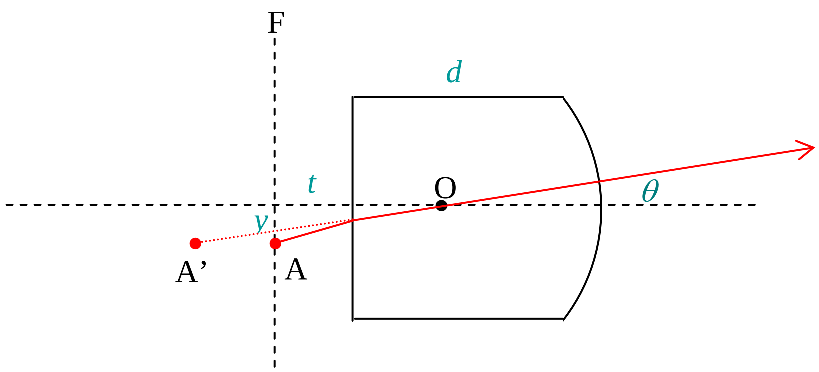
$$\frac{r}{r_f}=\frac{1\pm\sqrt{1-4(0.65)(0.35)}}{1.3}=1\,\text{or}\,\frac{7}{13}$$

$$r_f=1.86r \qquad \text{(1 point)}$$

Q3

利用最原始的单球面成像公式:

$$\frac{n_1}{S_O}+\frac{n_2}{S_I}=\frac{n_2-n_1}{R} \qquad (1)$$



(a)

<u>先考虑平面的成像</u>: $S_O=t$, (*t*是发光体到平面的距离)

若考虑发光体是在厚度为 T 的玻璃后面(上图没有给出),则 $S_O=t+T/n_G$,

n_G 是玻璃的折射率。平面的 R 为无穷大,利用式(1), $n_1=1$, $n_2=n$ (透镜

的折射率），得像距 $S_I = -n(t + T / n_G)$，像在平面的左边。此像为平面右边的球面的物，距离球面 $n(t + T / n_G) + d$。(1 point)

球面的成像：$S_O = n(t + T / n_G) + d$，$S_I = \infty$，$n_1 = n$，$n_2 = 1$，R为负数，利用式（1），得 $R = (n-1)(t + T / n_G + d / n)$ （2）(1 point)

$$t = \frac{1}{n-1} R - \frac{d}{n} - \frac{T}{n_G} \quad \text{(1 point)}$$

(b)

若 R 固定，则最大厚度 $d_{\max}$ 为（另式（2）中的 $t = 0$）$d_{\max} = \frac{nR}{(n-1)} - \frac{nT}{n_G}$。若膜板厚度小于此值，则可调节空气间隔 t 来使发光体位于焦平面上，大于此值则无法使发光体位于焦平面上。(1 point)

(c)

出射角：

如图，物 A 经平面成的像为 A'。从 A'射出的光经过球心 O 穿过球面无折射，由几何关系得 $\theta = \frac{y}{nt + Tn / n_G + (d - R)} = \frac{(n-1)y}{R}$。 （3）(3 points)

(d)

由 t 和 R 的偏差引起的误差可由式（3）微分后得

$$\frac{\delta a}{a} = \frac{\delta \theta}{\theta} = \frac{|\delta d| + |\delta R|}{nt + nT / n_G + (d - R)} \quad \text{（4）(2 points)}$$

Q4

(a)

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[3(\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{m} \right] = \frac{\mu_0}{4\pi} \frac{1}{R^3} \left[3\left((-m\hat{\mathbf{z}}) \cdot \hat{\mathbf{r}} \right) \hat{\mathbf{r}} - (-m\hat{\mathbf{z}}) \right]$$

$$= \frac{\mu_0}{4\pi} \frac{m}{R^3} \left[-3 \sin\alpha \hat{\mathbf{r}} + \left(\sin\alpha \hat{\mathbf{r}} - \cos\alpha \hat{\boldsymbol{\theta}} \right) \right] = \frac{\mu_0}{4\pi} \frac{m}{R^3} \left[-2 \sin\alpha \hat{\mathbf{r}} - \cos\alpha \hat{\boldsymbol{\theta}} \right]$$

From north to south, at an angle $\beta = \arctan(2 \tan\alpha) = 38.9^\circ$ pointing downwards. (2 points)

由北向南，与水平面成夹角 $\beta = \arctan(2 \tan\alpha) = 38.9^\circ$

(b)

$$\left| \vec{B} \right| \cong 0.5 \times 10^{-4} T \quad \text{(1 point)}$$

$$\vec{F} = I\vec{l} \times \vec{B} = IBl \sin\theta = 100 \times 10 \times 0.63 \times 5 \times 10^{-5} = 0.0315 N \quad \text{(2 points)}$$

Q5

No heat exchange during 1→2 and 3→4.　　过程 1→2、3→4 无热交换。 (2 points)

Heat absorbed in 2→3 2→3 过程吸热:

$$Q_h = \frac{3}{2}\left(P_2V_3 - P_2V_2\right) + P_2\left(V_3 - V_2\right) = \frac{5}{2}P_2\left(V_3 - V_2\right).\ \text{(2 points)}$$

Heat released in 4→1 4→1 过程放热:

$$Q_c = \frac{3}{2}\left(P_4V_1 - P_1V_1\right) = \frac{3}{2}\left(P_4 - P_1\right)V_1.\ \text{(2 points)}$$

$$e = 1 - \frac{Q_c}{Q_h} = 1 - \frac{3\left(P_4 - P_1\right)V_1/2}{5P_2\left(V_3 - V_2\right)/2} = 1 - \frac{3}{5}\frac{\left(P_4 - P_1\right)V_1}{P_2\left(V_3 - V_2\right)}$$

$$= 1 - \frac{3}{5}\frac{P_4 - P_1}{P_2}\frac{V_1}{V_3 - V_2} = 1 - \frac{3}{5}\left(\frac{P_4}{P_2} - \frac{P_1}{P_2}\right)\frac{V_1/V_2}{V_3/V_2 - 1}$$

$$= 1 - \frac{3}{5}\left(\left(\frac{V_3}{V_1}\right)^{5/3} - \left(\frac{V_2}{V_1}\right)^{5/3}\right)\frac{V_1/V_2}{V_3/V_2 - 1}$$

$$= 1 - \frac{3}{5}\left(\left(\frac{\alpha}{r}\right)^{5/3} - \left(\frac{1}{r}\right)^{5/3}\right)\frac{r}{\alpha - 1} = 1 - \frac{3}{5}\frac{\left(\alpha^{5/3} - 1\right)}{r^{2/3}\left(\alpha - 1\right)} \qquad \text{(2 points)}$$

Q6

First find the distance of CM from the center 首先求质心离盘心的距离:

$$y_{CM} = \frac{1}{M}\int_0^R \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \frac{M}{\pi R^2/2}\, y\,dx\,dy$$

$$= \frac{2}{\pi R^2}\int_0^R 2y\sqrt{R^2 - y^2}\,dy \ \text{(or start from here) (1 point)}$$

$$= \frac{2}{\pi R^2}\int_R^0 \sqrt{R^2 - y^2}\,d\left(R^2 - y^2\right) = \frac{2}{\pi R^2}\left.\frac{\left(R^2 - y^2\right)^{3/2}}{3/2}\right|_R^0 = \frac{4}{3\pi R^2}R^3 = \frac{4}{3\pi}R.\ \text{(1 point)}$$

The moment of inertia about the center is $\frac{1}{2}MR^2$. So the moment of inertia about CM is 相对于盘心的转动惯量为 $\frac{1}{2}MR^2$。因此，相对于质心的转动惯量为

$$I_{CM} = \frac{1}{2}MR^2 - M\left(\frac{4}{3\pi}R\right)^2 = \left(\frac{1}{2} - \frac{16}{9\pi^2}\right)MR^2.\ \text{(2 points)}$$

The moment of inertia about the point of contact is 相对于与地板的接触点的转动惯量为

$$I_C = \left(\frac{1}{2} - \frac{16}{9\pi^2}\right)MR^2 + Md^2,\ \text{(1 point)}$$

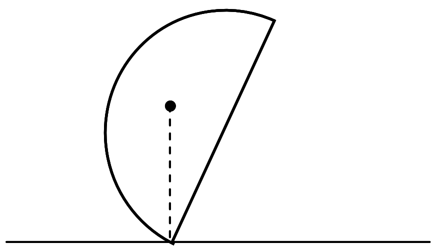
$d = \left(1 - \frac{4}{3\pi}\right)R = 0.5756R$.

Hence 因此

$$I_C = \left(\frac{1}{2} - \frac{16}{9\pi^2}\right)MR^2 + M\left(1 - \frac{4}{3\pi}\right)^2 R^2 = \left(\frac{3}{2} - \frac{8}{3\pi}\right)MR^2 = 0.65MR^2 \text{. (1 point)}$$

(b)

$I\omega = PR \Rightarrow \omega = \frac{PR}{I}$ (1 point)



$U_0 = Mgd$, (1 point)

$T_0 = \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{P^2R^2}{I}$, (1 point)

$U_f = Mg\sqrt{R^2 + d^2}$, $T_f = 0$ (1 point)

$U_0 + T_0 = U_f \Rightarrow$

$P^2 = 1.3M^2g\left(\sqrt{R^2 + d^2} - d\right) = 1.3M^2gR\left(\sqrt{1 + 0.5756^2} - 0.5756\right) = 0.752M^2gR$ (1 point)

Thus, 得 $P_{\min} = 0.867M\sqrt{gR}$ (1 point)

Part-2 卷-2

Q1

(a)

$N = \frac{1000}{235 \times m} = \frac{1000}{235 \times 1.67 \times 10^{-27}} = 2.548 \times 10^{27}$ (3 points)

(b)

$N = 1 + 3 + 9 + \cdots = \sum_{n=0}^{m} 3^n = \frac{1}{2}(3^{m+1} - 1)$ (5 points)

$m = \frac{\log(2N+1)}{\log 3} - 1 \approx 58$, $T = mt = 5.8 \times 10^{-7} s$ (2 points)

Q2

(a)

$\frac{hc}{2\pi L_p} = E_p = m_p c^2$ (1) (1 point)

$\frac{c^2}{L_p} = \frac{Gm_p}{L_p^2}$ (2) (2 points)

Solving (1) and (2) we get $L_p = \sqrt{\frac{Gh}{2\pi c^3}} = 1.6\times10^{-35} m = 1.6\times10^{-26} nm$ (1 point)

(b)

$$E_p = \frac{hc}{2\pi L_p} = \frac{1240(eV\cdot nm)}{2\pi\times1.6\times10^{-26}(nm)} = 1.23\times10^{28} eV \quad \text{(2 points)}$$

(c)

$$Z \equiv \frac{\lambda'-\lambda}{\lambda} = \frac{\omega}{\omega'} - 1 \quad \text{(1 point)}$$

$$(Z+1)^{-1} = \frac{\omega'}{\omega} = \gamma(1-\beta) = \sqrt{\frac{1-\beta}{1+\beta}}, \quad \text{(3 points)}$$

with Z = 1.03, we get $\beta = \frac{(Z+1)^2-1}{(Z+1)^2+1} = 0.61$. (2 points)

Away from us 离我们而去. (1 point)

(d)

$$D = v/H_0 = \frac{0.61\times3.0\times10^5}{21.7} MLY = 8.4\times10^9 LY \quad \text{(2 points)}$$

(e)

$$\Delta t = \frac{D}{c}\cdot\frac{\Delta E}{E_p} = 8.4\times10^9\times86400\times365\times\frac{3.0\times10^{10}}{1.23\times10^{28}} = 8.4\times8.64\times3.65\times\frac{3.0}{1.23} \quad \text{(4 points)}$$

$= 0.646\ s$

No. (1 point)

Q3

(a) According to the state equation of air in the adiabatic process 绝热过程(C_p/C_v = 7/5)

$$p_0 R_0^{3\times\frac{7}{5}} = p'(R+x)^{3\times\frac{7}{5}} \quad \text{(1 point)}$$

➔ $P' = P_0(1-\frac{21}{5}\frac{x}{R})$, that is $P_{out} - P_{in} = \frac{21}{5}\frac{x}{R}P_0$

➔ So the force 力为 $F = 4\pi R^2 \Delta P = -\frac{84}{5}\pi R P_0 x$ (1 point)

(b) The energy 势能为 $E = \int_0^{x_0} F dx = \frac{84}{5}\pi R P_0 \int_0^{x_0} x dx = \frac{42}{5}\pi R P_0 x_0^2$ (2 points)

(c) According to the continuity equation 利用水流连续性,

$$4\pi R^2 \frac{dx}{dt} = 4\pi r^2 v(r) \quad \text{(1 point)}$$

➔ Therefore 因此 $v(r)=\frac{R^2}{r^2}\frac{dx}{dt}$

(d) The kinetic energy of water 水的动能为

$$K=\int_R^\infty \frac{1}{2}(4\pi r^2\rho dr)(\frac{R^2}{r^2}\frac{dx}{dt})^2=2\pi R^3\rho(\frac{dx}{dt})^2 \quad \text{(2 points)}$$

(e) $\omega=\sqrt{\frac{\frac{42}{5}\pi RP_0}{2\pi R^3\rho}}=\frac{1}{R}\sqrt{\frac{21P_0}{5\rho}}$ (1 point)

(f) Note that a membrane has two surfaces, so its surface tension energy is $E=2\gamma a^2$
薄膜有两个表面，因此表面能量为 $E=2\gamma a^2$ (1 point)

Let one side increase by dx, the energy change is $dE=2\gamma a dx$, so $F_{tension}=2\gamma a$.令其中一边外移一小段 dx，则能量的改变为 $dE=2\gamma a dx$，因此 $F_{tension}=2\gamma a$ (1 point)

(g) $E=4\gamma\pi R^2$, (1 point)

(h) ➔ $8\gamma\pi RdR=dE=PdV=P(4\pi R^2dR)$ ➔ $P_{tension}=2\gamma/R$. (1 point)

平衡时 $P_0=P_{gas}=P_{atm}+P_{tension}=P_{atm}+\frac{2\gamma}{R}$ at equilibrium. (1 point)

The net change of pressure when $R->R+x$ is
当 $R->R+x$ 时总的压强的变化为

$$dP=dP_{gas}-dP_{tension}=-3\kappa P_0\frac{x}{R}+\frac{2\gamma x}{R^2} \quad \text{(2 points)}$$

$$dE=dP\cdot 4\pi R^2x=-4\pi\left(3\kappa P_0R-2\gamma\right)x \quad \text{(1 point)}$$

$E=2\pi\left(3\kappa P_0R-2\gamma\right)x_0^2$, $\kappa=\frac{7}{5}$, so $E=\pi\left(\frac{42}{5}P_0R-4\gamma\right)x_0^2$ (1 point)

$$\omega=\sqrt{\frac{\frac{42}{5}\pi RP_0-4\pi\gamma}{2\pi R^3\rho}}=\frac{1}{R}\sqrt{\frac{1}{\rho}\left(\frac{21P_0}{5}-\frac{2\gamma}{R}\right)} \quad \text{(1 point)}$$