Part-I

Q1 (**9 points**)

Rotational inertia of the beam 杆的转动惯量的一般表达式: $I(l_1,l_2)=\int_{l_1}^{l_2} l^2 m/L dl = \frac{m}{3L} l^3 \Big|_{l_1}^{l_2}$.

a) $I = I(0,L) = \frac{m}{3}L^2$

Conservation of angular momentum 角动量守恒:

$I\omega + mL^2\omega = mLv_0 \quad \Rightarrow \quad \omega = \frac{Lmv_0}{I+L^2m} = \frac{3v_0}{4L}$.

Energy lost 动能损失:

$\Delta E = \frac{1}{2}mv_0^2 - \left(\frac{1}{2}I\omega^2 + \frac{1}{2}mL^2\omega^2\right) = mv_0^2/8$. (3 points)

b) $I = I(-\frac{3}{4}L, \frac{1}{4}L) = \frac{7}{48}mL^2$, or $I = \frac{1}{12}mL^2 + m\left(\frac{L}{4}\right)^2 = \left(\frac{1}{12}+\frac{1}{16}\right)mL^2 = \frac{7}{48}mL^2$

Conservation of momentum and angular momentum 角动量、动量守恒:

$$\begin{cases} mv_0 = 2mv', \\ I\omega + m\left(\frac{L}{4}\right)^2\omega = m\frac{L}{4}v_0 \end{cases} \Rightarrow \begin{cases} \omega = \frac{6v_0}{5L}, \\ v' = v_0/2 \end{cases}.$$

Energy lost 动能损失:

$\Delta E = \frac{1}{2}mv_0^2 - \left(\frac{1}{2}I\omega^2 + \frac{1}{2}m\left(\frac{L}{4}\right)^2\omega^2 + mv'^2\right) = mv_0^2/10$. (3 points)

c) $I = I(-\frac{1}{2}L, \frac{1}{2}L) = \frac{1}{12}mL^2$

Conservation of momentum, angular momentum and energy 角动量、动量、能量守恒:

$$\begin{cases} mv_0 = mv_1 + mv_2 \\ I\omega + m\frac{L}{2}v_2 = m\frac{L}{2}v_0 \\ \frac{1}{2}I\omega^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}mv_1^2 = \frac{1}{2}mv_0^2 \end{cases} \Rightarrow \begin{cases} \omega = \frac{12v_0}{5L} \\ v_1 = \frac{2v_0}{5} \\ v_2 = \frac{3v_0}{5} \end{cases} \quad \text{(3 points)}$$

Q2 (**10 points**)

a) $m_1\ddot{\vec{r}}_1 = G\frac{m_1m_2}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_2-\vec{r}_1), \quad m_2\ddot{\vec{r}}_2 = G\frac{m_1m_2}{|\vec{r}_1-\vec{r}_2|^3}(\vec{r}_1-\vec{r}_2)$. (1 point)

b) $\vec{r}_1 = \frac{m_2}{m_1+m_2}\vec{r} + \vec{r}_c, \quad \vec{r}_2 = \frac{m_1}{m_1+m_2}\vec{r} + \vec{r}_c$.

$$\begin{cases}(m_1+m_2)\ddot{\vec{r}}_c=0 \quad\Rightarrow\quad \ddot{\vec{r}}_c=0\\ \dfrac{m_1m_2}{m_1+m_2}\ddot{\vec{r}}=-G\dfrac{m_1m_2}{|\vec{r}|^3}\vec{r} \quad\Rightarrow\quad \ddot{\vec{r}}=-G\dfrac{m_1+m_2}{|\vec{r}|^3}\vec{r}\end{cases} \qquad \text{(2 points)}$$

c) $\vec{r}_c=0$. (1 point)

d) The equation for $\vec{r}$ is the same as a uniform circular motion，$\vec{r}$ 满足的方程与匀速圆周运动满足的方程一致. Therefore 因此,

$$\frac{G(m_1+m_2)}{a^2}=\omega^2 a \quad\Rightarrow\quad \omega=\frac{\sqrt{G(m_1+m_2)}}{a^{3/2}}.$$

$\vec{r}(t)=a\cos(\omega t)\vec{x}_0+a\sin(\omega t)\vec{y}_0$. (3 points)

e) $m_1=m_2=m$,

$$T=\frac{2\pi a^{3/2}}{\sqrt{2Gm}} \quad\Rightarrow\quad a=\left(\frac{2T^2Gm}{4\pi^2}\right)^{1/3}=\left(\frac{9\times10^8\times6.7\times10^{-11}\times4\times10^{30}}{4\pi^2}\right)^{1/3}=\left(6.116\times10^{27}\right)^{1/3}=1.8\times10^9\ \text{m}$$. (3 points)

Q3 (**12 points**)

a) $q'=-QR/x,\quad x'=R^2/x.$

$$f=-\frac{1}{4\pi\varepsilon_0}\frac{Qq'}{(x-x')^2}=\frac{1}{4\pi\varepsilon_0}\frac{Q^2Rx}{(x^2-R^2)^2}$$

$$W=\int_0^L f(x)dx=\frac{1}{8\pi\varepsilon_0}\frac{L^2Q^2}{R(R^2-L^2)}.$$ (3 points) 不用积分扣 2 分。

b) Since the outside charge does not create any field inside the conductor sphere, the work done here is no different from the one in a):外电荷对导体内部无影响，所以结果与 a)一样。

$$W=\frac{1}{8\pi\varepsilon_0}\frac{L^2e^2}{R(R^2-L^2)}.$$ (2 points)

c) The static electric field on the charge $q_2=LQ/R$ outside the conducting sphere is equivalent to a field generated by three point charges: $q_1=Q$ at $(0,0)$, image charge $q_2'=-q_2R/x$ at $(l'=R^2/l,0)$, and $q_2''=-q_2'$ at $(0,0)$. Therefore

球外 $q_2=LQ/R$ 受的静电力来自以下三个点电荷: $q_1=Q$ 在 $(0,0)$，镜像电荷 $q_2'=-q_2R/x$ 在 $(l'=R^2/l,0)$，以及 $q_2''=-q_2'$ 在 $(0,0)$。

$$f(x)=\frac{1}{4\pi\epsilon_0}\left[\frac{q_2q_2'}{(x-x')^2}+\frac{q_2(q_1+q_2'')}{x^2}\right]=\frac{1}{4\pi\epsilon_0}\left[-\frac{q_2^2xR}{(x^2-R^2)^2}+\frac{q_2(xq_1+Rq_2)}{x^3}\right].$$ (2 points)

And the work done by electric field is then 电场做功

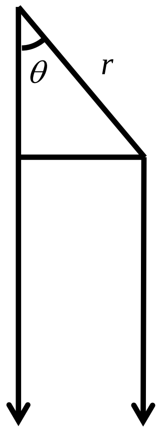
$$\int_{\infty}^{R^2/L} f(x)dx = \frac{1}{8\pi\varepsilon_0}\left[\frac{L^2q_2^2}{R^3-L^2R} - \frac{Lq_2(Lq_2+2Rq_1)}{R^3}\right]$$

$$= \frac{1}{8\pi\varepsilon_0}\left[\frac{L^4Q^2}{R^5-R^3L^2} + \frac{L^2(2R^2-L^2)Q^2}{R^5}\right] = \underline{\frac{1}{8\pi\varepsilon_0}\frac{L^2(L^4-2L^2R^2+2R^4)Q^2}{R^5(R^2-L^2)}} \quad \text{(3 points)}$$

d) $W = W_{\text{in c)}}$ (2 points)

Q4 (**9 points**)

a) $\Delta t = \frac{r}{v} - \frac{r\cos\theta}{c}$



$$\underline{\tilde{v} = \frac{r\sin\theta}{\Delta t} = \frac{cv\sin\theta}{c - v\cos\theta}} \quad \text{(2 points)}$$

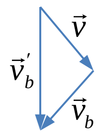
$$\underline{\tilde{v} = \frac{0.9\cdot 0.707c}{1-0.9\cdot 0.707} = 1.75c}. \text{ (1 point)}$$

b) $\underline{\tilde{v} = \frac{v_s v\sin\theta}{v_s - v\cos\theta} < 0 \quad \Rightarrow \quad v_s - v\cos\theta < 0}$ (2 points)

Thus $\underline{v > v_s/\cos\theta}$. (1 point)

c) Let the emission angle of the beads be θ'. Then to ensure the net velocity of the beads is along the Y-direction, we have 令小球的发射角为θ'，为使球的合速度沿 Y-方向，须有

$$v\sin\theta = v_b\sin\theta', \quad v_b' = v\cos\theta + v_b\cos\theta' = v\cos\theta + v_b\sqrt{1-(v/v_b)^2} > v\cos\theta$$



$$\Delta t = \frac{r}{v} - \frac{r\cos\theta}{v_b'}.$$

$$\underline{\tilde{v} = \frac{v\sin\theta(v\cos\theta + v_b\sqrt{1-(v/v_b)^2})}{v_b\sqrt{1-(v/v_b)^2}} > 0} \quad \text{(2 points)}$$

<u>So it will not happen 不会发生</u>. (1 point)

Q5

(a) The initial number of mole of air in the tire is 轮胎内原有气体摩尔量 $n_i = \frac{P_iV_0}{RT_a}$

The final number of mole of air in the tire is 打完气轮胎内有气体摩尔量 $n_f = \frac{P_fV_0}{RT_a}$

So number of mole of air pumped into the tire is 因此，打进轮胎的气体摩尔量为

$$n_f - n_i = \frac{\left(P_f - P_i\right)V_0}{RT_a}\text{。}$$

Inside the compressor, this amount of air has volume 在压缩机里，此摩尔量的气体的体积为

$$V' = \frac{\left(n_f - n_i\right)RT_a}{P_c} = \frac{P_f - P_i}{P_c}V_0$$

The work done by the compressor is hence 压缩机做功 $\underline{W_c = P_c V' = \left(P_f - P_i\right)V_0}$ (1 point)

The internal energy of the gas now in the tire increases by 轮胎内气体内能增加为

$$\underline{\Delta E = (n_f - n_i)C_V T_a + W_c = \frac{R}{\gamma - 1}(n_f - n_i)T_a + \left(P_f - P_i\right)V_0} \qquad \text{(1 point)}$$

The maximum temperature is then 因此这时的温度

$$T_{max} = T_a + \frac{W_c}{n_f C_V} = T_a + \frac{\left(P_f - P_i\right)V_0}{\frac{P_f V_0}{RT_a}\frac{R}{\gamma - 1}} = \left[\gamma - (\gamma - 1)\frac{P_i}{P_f}\right]T_a$$

The maximum pressure is 气压

$$P_{max} = \frac{n_f RT_{max}}{V_0} = \left[\gamma - (\gamma - 1)\frac{P_i}{P_f}\right]P_f = \gamma P_f - (\gamma - 1)P_i$$

The minimum P_c required is 最小 P_c 必须满足

$$\underline{P_c \geq P_{max} = \gamma P_f - \left(\gamma - 1\right)P_i} \qquad \text{(2 points)}$$

(b)

The total number of stokes is 打气总次数 $\underline{N = \frac{n_f - n_i}{P_a V_p / RT_a} = \frac{\left(P_f - P_i\right)V_0}{P_a V_p}}$. (1 point)

During the *j*-th stroke, in the adiabatic compression inside the pump from P_a to P_j 第 j 个斯托克循环绝热压缩使压强从 P_a 变为 P_j

$$P_a V_p^{\gamma} = P_j V'^{\gamma}$$

where *V'* is the volume of the air inside the pump after the adiabatic compression. *V'*：压缩后气体体积
The internal energy of the air at this moment is 此时气体的内能

$$\frac{1}{\gamma - 1}P_j V' = \frac{1}{\gamma - 1}P_j\left(\frac{P_a}{P_j}\right)^{\frac{1}{\gamma}}V_p$$

The amount of work done to inject this amount of air into the tire is 打气所做的总功

$$P_j V' = P_j\left(\frac{P_a}{P_j}\right)^{\frac{1}{\gamma}}V_p$$

Hence the change in internal energy of the air inside the tire during the *j*-th stroke is 内能的变化

$$\underline{\Delta U = \frac{1}{\gamma - 1}P_j V' + P_j V' = \frac{\gamma}{\gamma - 1}P_j V' = \frac{\gamma}{\gamma - 1}P_j\left(\frac{P_a}{P_j}\right)^{\frac{1}{\gamma}}V_p} \quad \text{(2 points)}$$

On the other hand, 另一方面

$$\Delta U=\Delta\left(\frac{PV}{\gamma-1}\right)=\frac{1}{\gamma-1}\left(P_j-P_{j-1}\right)V_0=\frac{1}{\gamma-1}V_0\Delta P=\frac{\gamma}{\gamma-1}V_pP_a^{\frac{1}{\gamma}}P_j^{\frac{\gamma-1}{\gamma}}$$

So 因此

$$P_j^{\frac{1-\gamma}{\gamma}}\Delta P=\gamma\frac{V_p}{V_0}P_a^{\frac{1}{\gamma}}$$

Replacing the finite-difference equation by differential equation and integrate 改用微分形式表示

$$P^{\frac{1-\gamma}{\gamma}}dP=\gamma\frac{V_p}{V_0}P_a^{\frac{1}{\gamma}}dj$$

$$\int_{P_i}^{P_{max}}P^{\frac{1-\gamma}{\gamma}}dP=\int_0^N\gamma\frac{V_p}{V_0}P_a^{\frac{1}{\gamma}}dj$$

We have 我们得到

$$\gamma\left(P_{max}^{\frac{1}{\gamma}}-P_i^{\frac{1}{\gamma}}\right)=\gamma\frac{V_p}{V_0}P_a^{\frac{1}{\gamma}}N$$

$$P_{max}^{\frac{1}{\gamma}}-P_i^{\frac{1}{\gamma}}=P_a^{\frac{1-\gamma}{\gamma}}\left(P_f-P_i\right)$$

$$\underline{P_{max}=P_i\left[1+\left(\frac{P_a}{P_i}\right)^{\frac{1}{\gamma}}\frac{P_f-P_i}{P_a}\right]^{\gamma}}\quad\text{(3 points)}$$

Part-II

Q1 **(20 points)**

a) $2\pi rB = \mu_0 J\pi r^2 \Rightarrow B = \mu_0 Jr/2$

Energy per length is 单位长度能量 $W = \frac{1}{\mu_0}\int_0^R B^2 2\pi r dr = \frac{\mu_0 I^2}{8\pi}$

Compare with 相比较 $\underline{W = \frac{1}{2}LI^2 \Rightarrow L = \frac{\mu_0}{4\pi}}$ (H/m) (1 point)

b) Newton's second law gives us the differential equation 牛顿第二定律引出的方程

$$m\ddot{\vec{x}} = -e\vec{E} - m\frac{\dot{\vec{x}}}{\tau} \quad \text{(1 point)}$$

c)

$$m\ddot{\vec{x}} = -e\vec{E}_0 e^{i\omega t} - m\frac{\dot{\vec{x}}}{\tau}$$

$$i\omega m\dot{\vec{x}} = -e\vec{E}_0 - m\frac{\dot{\vec{x}}}{\tau}$$

$$\underline{\dot{x} = \frac{-e\tau E_0}{(1+i\omega\tau)m}} \quad \text{(2 points)}$$

d) The current density is given by 电流密度 $\underline{J = -ne\dot{x} = \frac{e^2\tau n E_0}{(1+i\omega\tau)m}}$ (2 points)

e) $\frac{I}{\pi r^2} = \frac{e^2\tau n}{(1+i\omega\tau)m}\frac{V}{D}$, so $\underline{V = \frac{D}{\pi r^2}\frac{(1+i\omega\tau)m}{e^2\tau n}I}$ (2 points)

f) The real part of the impedance represents the resistance 阻抗实部 $\underline{R = \frac{D}{\pi r^2}\frac{m}{e^2\tau n}}$ (1 point)

The imaginary part represents 虚部 $\omega L_I = \frac{D}{\pi r^2}\frac{m\omega}{e^2 n}$, so $\underline{L_I = \frac{D}{\pi r^2}\frac{m}{e^2 n}}$. (1 point)

g) $\underline{L_I = R\tau = 2.0\times 10^{-9}(H)}$. $\underline{L_{Faraday} = \frac{\mu_0}{4\pi}D = 1.0\cdot 10^{-7}\cdot 10^{-6} = 1.0\cdot 10^{-13}(H)}$ (2 points)

h)

$$V_m - V_{m-1} = \frac{Q}{C} = \frac{I_m}{i\omega C}$$

$$V_m = i\omega L K_m,\ V_{m-1} = i\omega L K_{m-1}$$

$$\begin{cases} I_{m+1} + K_m = I_m \\ I_m + K_{m-1} = I_{m-1} \end{cases}$$

$$\underline{-i\omega C\left(V_{m+1}-2V_m-V_{m-1}\right)+\frac{V_m}{\omega^2 LC}=0}$$ (5 points)

i)

$$e^{-ika}+e^{ika}-2+\frac{1}{\omega^2 LC}=0$$

$$\underline{\omega=\sqrt{\frac{1}{4LC}}\left(\sin\frac{ka}{2}\right)^{-1}}$$ (2 points)

j) Yes 是(1 point)

Q2 (**30 points**)

a) Largest 最大: compressed 压缩; Smallest 最小: stretched 伸展, or vice versa 反之亦然 (1 point)

b) L_G: Joule/s → N · m/s → kg · m^{2}s^{-3}

RHS: $A^a \mathrm{B^b kg^2 m^4 s^{-6}}$

$\underline{\Rightarrow\ A^a=c^{-5}, B^b=G}$ (1 point)

$\underline{\Rightarrow\ L_G=c^{-5}GM^2R^4\omega^6}$ (1 point)

c)

$$G\frac{M_1M_2}{R^2}=M_1\omega^2R_1=M_2\omega^2R_2$$

with $R_1+R_2=R$

$$\Longrightarrow\ R^3=\frac{G(M_1+M_2)}{\omega^2}$$

Kinetic energy: $\bar{M}\equiv\frac{M_1M_2}{M_1+M_2}$, $M\equiv M_1+M_2$

$$\underline{KE=\frac{1}{2}M_1\omega^2R_1^2+\frac{1}{2}M_2\omega^2R_2^2=\frac{1}{2}\frac{\bar{M}GM}{R}=\frac{1}{2}(2\pi)^{2/3}\bar{M}(GM)^{2/3}T^{-2/3}}$$ (2 points)

d) Neutron binary 双中子星系统

$\frac{dE_k}{dT}=-\frac{1}{3}(2\pi)^{2/3}\bar{M}(GM)^{2/3}T^{-5/3}$, (1 point)

$L_G=\frac{dE_k}{dt}=\frac{dE_k}{dT}\frac{dT}{dt}$ (1 point)

$-\frac{dT}{dt}=3G^{\frac{5}{3}}c^{-5}M_1M_2(M_1+M_2)^{-\frac{1}{3}}(2\pi)^{\frac{8}{3}}T^{-\frac{5}{3}}$ (1 point)

$$-\frac{dT}{dt}=1.4\times10^{-67}\cdot4\cdot10^{60}\cdot(2)^{-\frac{1}{3}}(3\cdot10^{4})^{-\frac{5}{3}}=1.4\times10^{-67}\cdot4\cdot10^{60}\cdot0.79\cdot3.4\cdot10^{-8}=1.5\times10^{-14}$$ (1 point)

e) For Sun-Earth system,日-地系统 with $\Delta T = 1s$

$$-\frac{dT}{dt}=1.4\times10^{-67}\cdot2\cdot10^{30}\cdot6\cdot10^{24}(365.24\cdot86400)^{-\frac{5}{3}}=1.7\cdot10^{-12}\cdot3.2\cdot10^{-13}=5.4\cdot10^{-25}$$ (1 point)

$$t=\frac{dt}{dT}=1.8\cdot10^{25}s=\frac{1.8\cdot10^{25}}{365.24\cdot86400}=\frac{1.8\cdot10^{25}}{3.155\cdot10^{6}}y=5.7\cdot10^{18}y$$. (1 point)

f) $R_s = \frac{MG}{c^2}$ (1 point)

g) $R_s = GMc^{-2}$, or $R_s c^2 G^{-1} = M$. (1 point)

Also $v = \omega R/2$. (1 point)

$$L_G = Gc^{-5}\bar{M}^2R^4\omega^6 = 16Gc^{-5}c^4G^{-2}R_s^2R_s^{-2}v^6 = 16\frac{c^5}{G}\left(\frac{v}{c}\right)^6 \approx 16\frac{c^5}{G}$$ (1 point) 数量级正确即可

$$=16\frac{3^5}{6.7}\cdot10^{40}\cdot10^{11}=5.9\cdot10^{53}W$$ (Order of magnitude only)

h) $L_0 = 2R, L' = \pi R \quad \Longrightarrow \quad \varepsilon \approx 1$ (2 points)

i) $(R'/R_s)^2 = 1/\varepsilon'^2$ with $R' = 10^4$light year $= 9.46\times10^{19}m, R_s = \frac{M_S G}{c^2} = 1.49\times10^3 m$

$$\varepsilon' = \varepsilon R_s / R' = 1.5\cdot1.5\cdot10^3\cdot10^{-29}/0.95 = 2.4\cdot10^{-22}$$. (3 points)

j) $\lambda_1 = 2L = 6\text{m}, f_1 = \frac{v}{\lambda_1} = 1067\text{Hz}$ (2 points)

k) 1st resonance: 第一个共振态

$$\frac{1}{2}m\omega^2x^2=\frac{1}{2}k_BT \Rightarrow x=\left(\frac{k_BT}{m\omega^2}\right)^{1/2}=\left(\frac{42\cdot1.4\cdot10^{-24}}{1.1\cdot10^3}\right)^{1/2}\frac{1}{2\pi\cdot1.07\cdot10^3}=3.3\cdot10^{-17}m$$ (4 points)

l) The change in the length due to gravitational wave should be at least in the same order of magnitude as 10^{-15}m 重力波所引起的长度改变量至少为10^{-15}m量级

$$\frac{x}{L}=\frac{R_s}{D}\Rightarrow D=\frac{R_sL}{x}=\frac{9\cdot10^3}{3.3\cdot10^{-17}}=2.7\cdot10^{20}m=\frac{2.7\cdot10^{20}}{9.5\cdot10^{15}}=2.8\cdot10^4Ly$$ (2 points)

m) $$I=\frac{10^{-6}L_G}{4\pi D^2}=\frac{5.9\cdot10^{53}\cdot10^{-6}}{4\pi\cdot2.7^2\cdot10^{40}}=6.9\cdot10^5(W/m^2)$$. We would be toast. (2 points)